Dimensional reduction in superconducting arrays and frustrated magnets

International Workshop on Frustrated Magnetism, September 2004

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When do two- and three-dimensional materials show exotic physics familiar from one spatial dimension?

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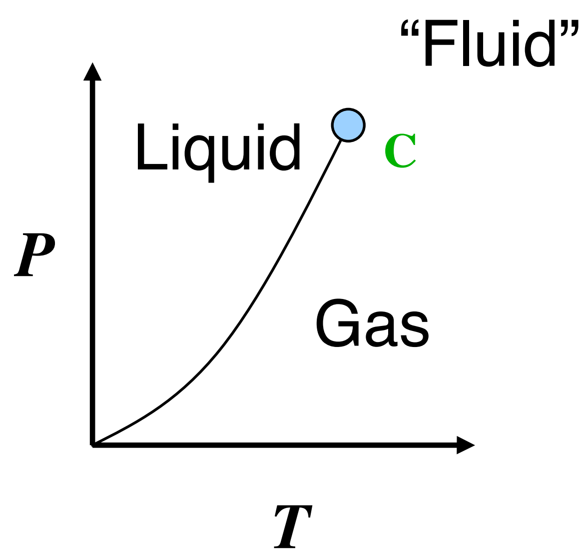
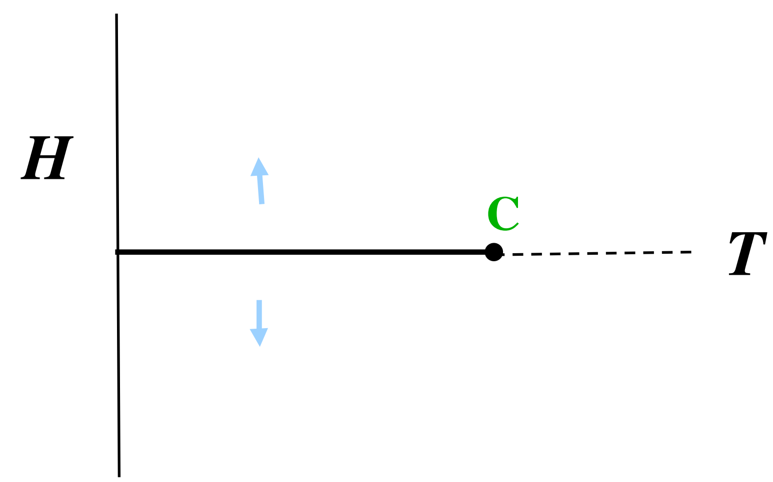
Outline

- **Part I**: Some materials are naturally modeled by local spin interactions involving >2 spins (e.g., ring exchange).
- **Part II**: Such interactions, when two-spin interactions are absent, can effectively reduce the dimensionality of an “isotropic” system.
- (Examples of familiar problems that “mix” one and two dimensions)
- **T**-breaking superconductivity and Sr_2RuO_4: when is there global **T** order in a Josephson-junction array of **T**-breaking superconductors?
- Similar behavior in frustrated magnetism (2D and 3D pyrochlores)
- Exact self-dualities and dimensional reduction in quantum models (Conservation laws and relation to “Kitaev model”)

Why dimensionality matters, I





Ising (uniaxial) ferromagnet

$$\rho_L - \rho_G \sim \left(\frac{T_C - T}{T_C}\right)^{\beta}$$

$$M_{\uparrow} - M_{\downarrow} \sim \left(\frac{T_C - T}{T_C}\right)^{\beta}$$

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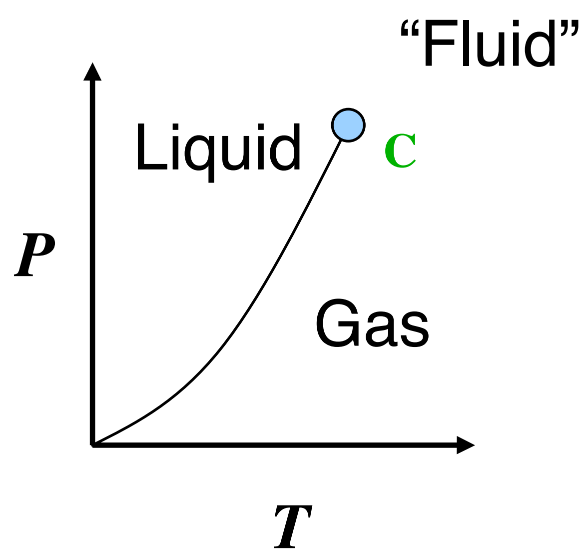

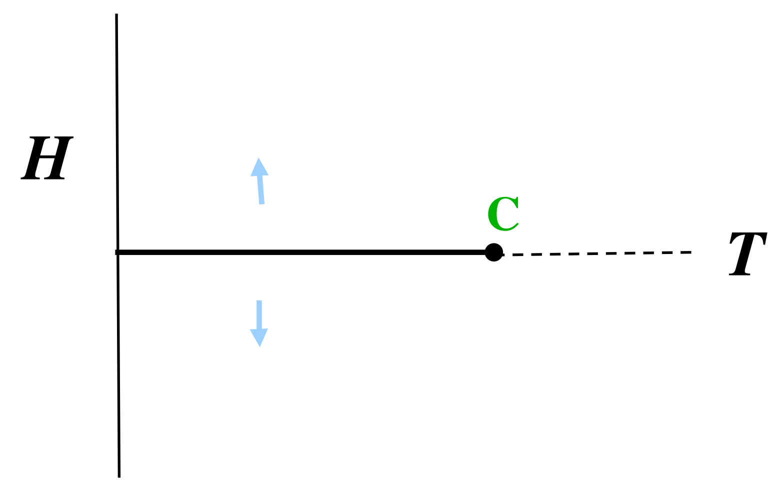

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$$M_\uparrow - M_\downarrow \sim \left(\frac{T_C - T}{T_C}\right)^{\beta}$$

$$\text{Experiment}: \quad \beta = 0.322 \pm 0.005$$

$$\text{Theory}: \quad \beta = 0.325 \pm 0.002$$

Dimensionality in quantum magnetism

Universal quantities like critical exponents are largely determined by **symmetry** and **dimensionality**.

At zero temperature, many models show quantum phase transitions between different ground states.

Frequently a quantum phase transition in **d** dimensions is in the same "universality class" as a **classical** transition in **d+1** dimensions.

An example is the transverse-field Ising model (S=1/2):

$$H = -K \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x$$

This model's transition at *K*=*h* is of the **2D classical Ising type.**

Dimensionality in quantum magnetism, II

Materials in one dimension show exotic physics that is much more difficult to observe in d > 1.

An example is the quantum Heisenberg model in one dimension:

$$H = J \sum_i \mathbf{s}_i \cdot \mathbf{s}_{i+1}, \quad J > 0$$

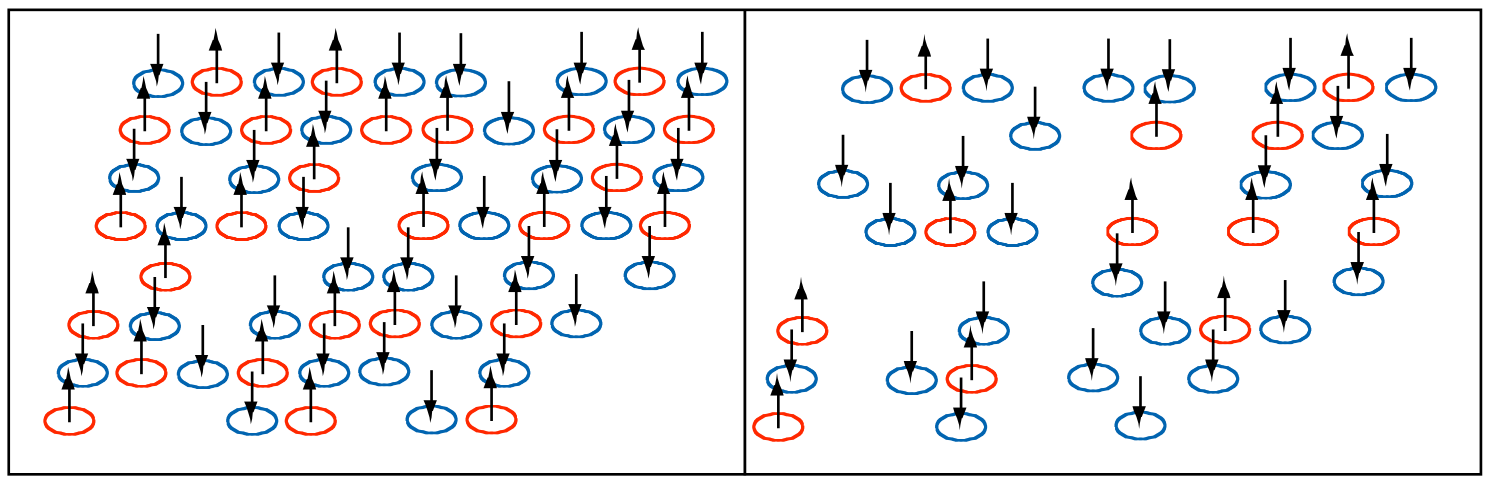
For S=1/2, this model has a quantum critical ground state (power-law correlations) whose elementary excitations are “spinons” of spin 1/2.

This is very different from either the 1D ferromagnet or 2D antiferromagnet, which have long-range order and “magnon” (spin-wave) elementary excitations of spin 1.

Are there quantum models that show behavior “between” 1D and 2D?

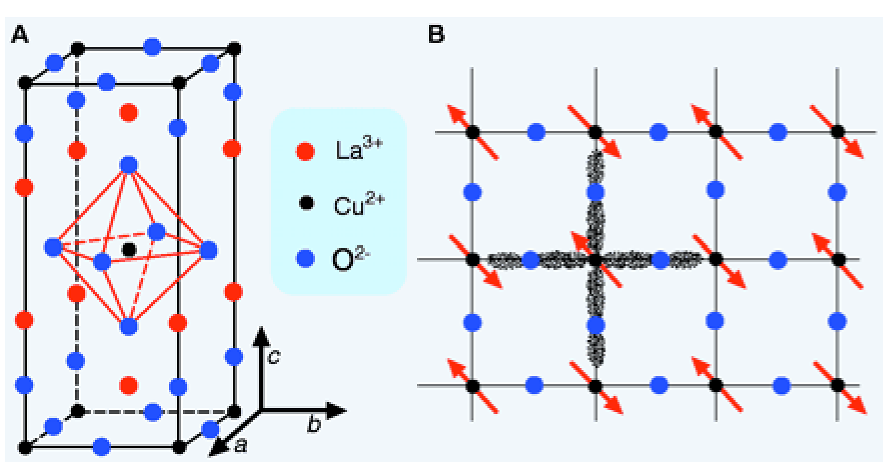
Real materials between 1D and 2D
Diluted quantum magnets

x=0.1 Occupied sites percolate | x=0.9 Vacancies percolate



Percolation threshold for square lattice: 40.7%

$La_2Cu_x(Zn,Mg)_{1-x}O_4$

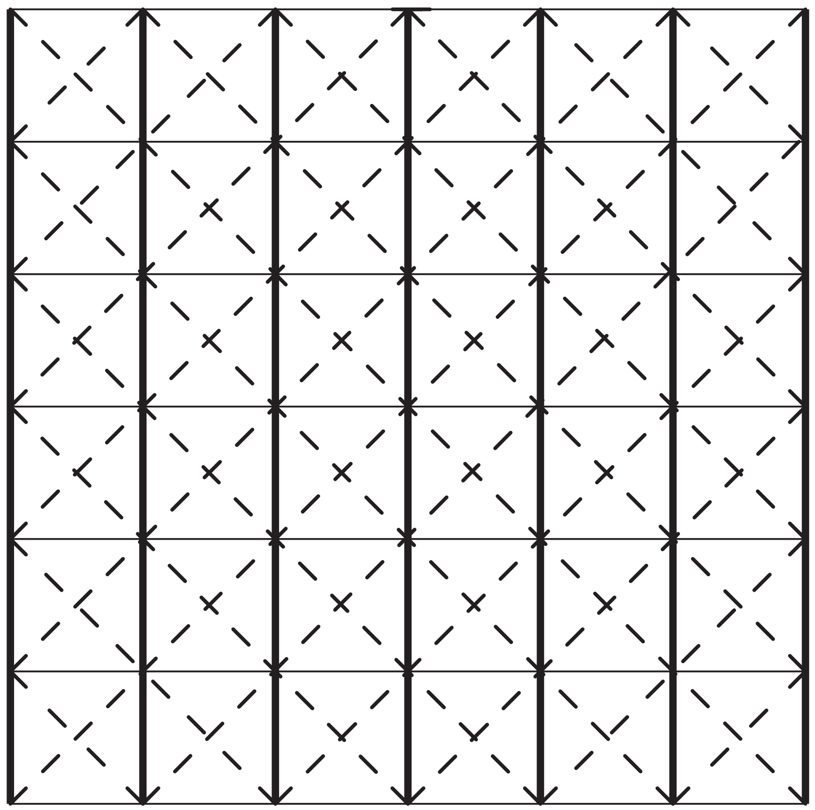


Static dilution of a 2D S=1/2 Heisenberg AF
(Vajk et al., *Science* 2002)

At threshold, connected cluster is a random fractal of dimension 91/48. Order and stiffness depend on competition between 1D and 2D physics.

Here **lack** of frustration is key.

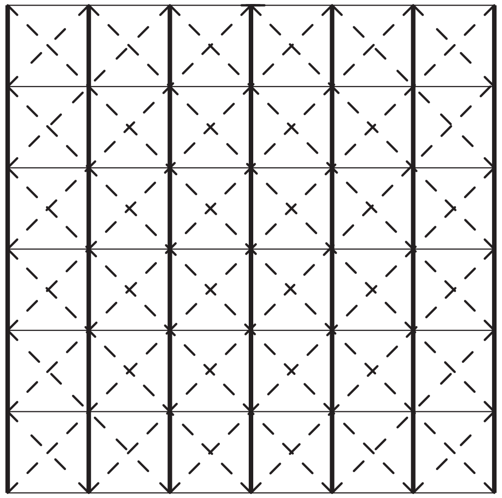
Models between 1D and 2D, cont'd



Anisotropic models (coupled chains)

Nersesyan & Tsvelik
Starykh & Balents
Moukouri

Note:



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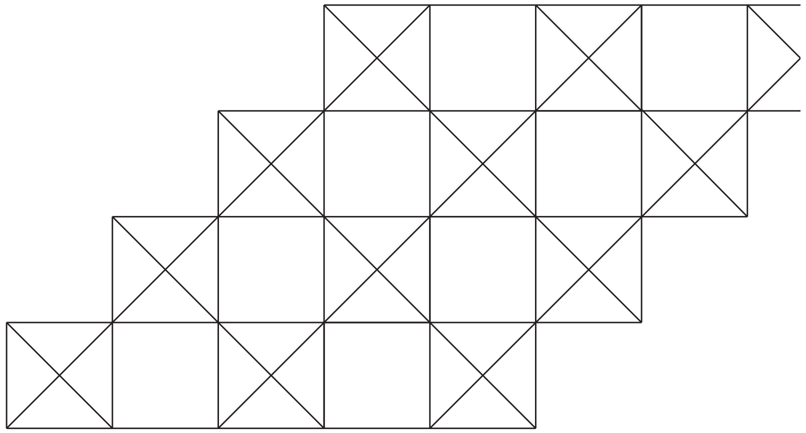
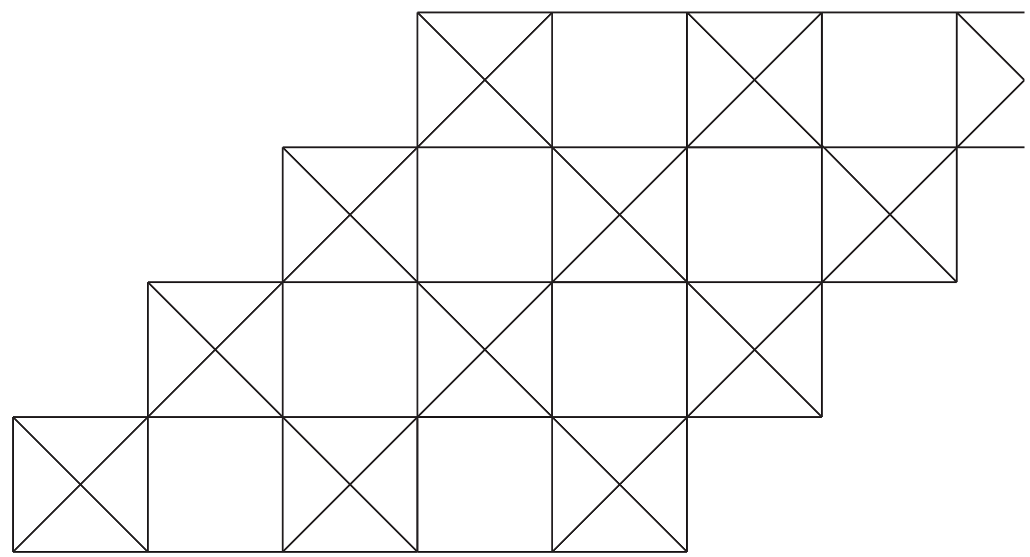


“**isotropic**” 2D model pyrochlore
(Henley, Tchernyshyov, Moessner et al.)

Effective Hamiltonians for frustrated magnets: many-spin interactions in 2D model pyrochlore



Infinitely degenerate set of classical ground states.
1/S quantum fluctuations select out collinear states
with effective energy for Ising spins

$$H = -K \sum_{\Box} s_{\Box}^{1} s_{\Box}^{2} s_{\Box}^{3} s_{\Box}^{4}, \quad s = \pm 1$$

and a constraint: tetrahedra have two +1 and two -1 spins.
Result: **one-dimensional** family of ground states (Henley, Tchernyshyov, Moessner, et al.) but two-dimensional finite-*T* phase transition.

T-breaking superconductivity

Familiar superconductors break a continuous *U(1)* symmetry at the superconducting phase transition: this *U(1)* corresponds to the SC phase.

Magnets, on the other hand, break time-reversal symmetry (**T**).

We now know of several superconducting phases that break **T**: when the Cooper pair wavefunction is complex, it and its conjugate are degenerate superconducting states related by **T**.

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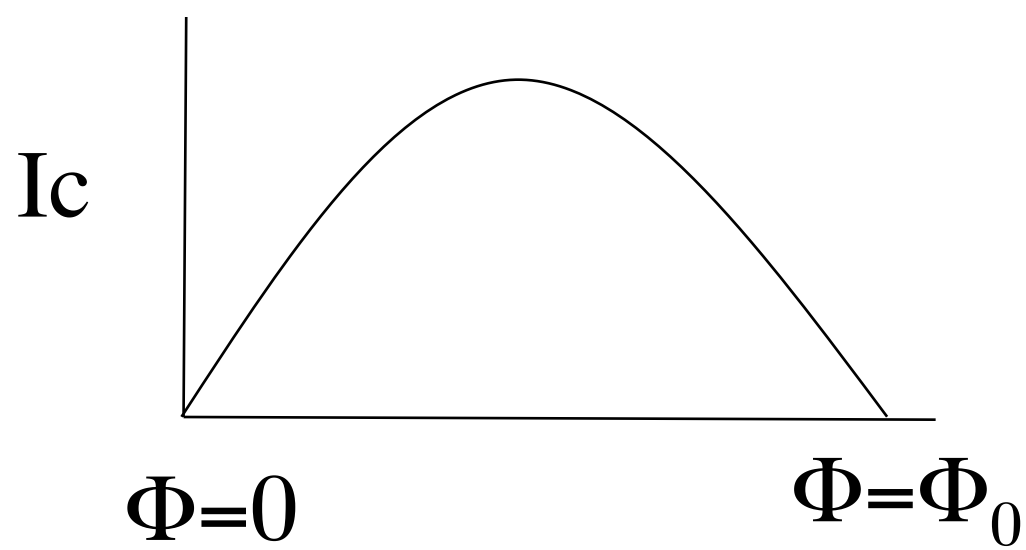
The standard example is the A phase of superfluid helium-3 (a *p+ip* superconductor). Other examples are:

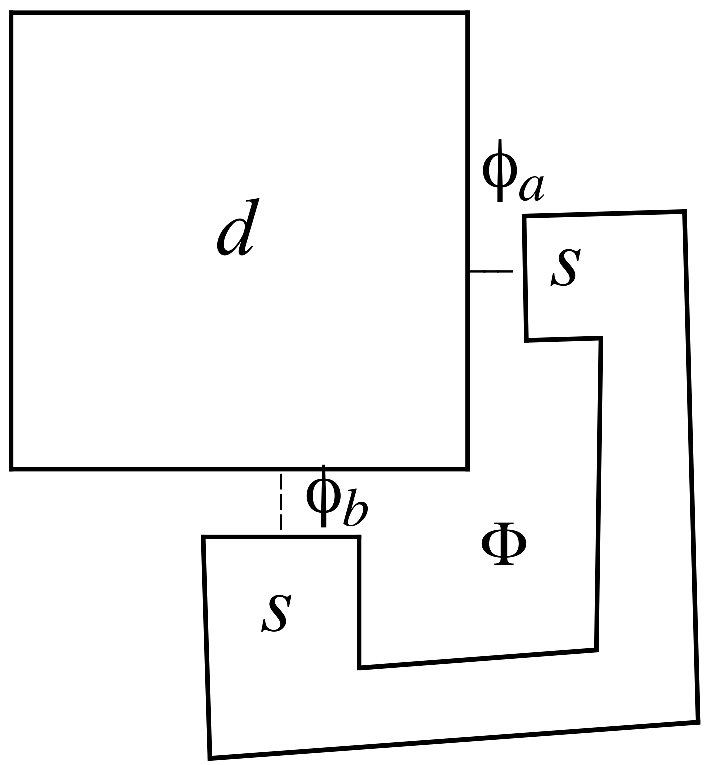
likely *p+ip* ordering in Sr_2RuO_4 , but some experimental questions
FQHE states of composite fermions (Moore-Read, Haldane-Rezayi)

RVB theories predict *d+id* ordering in hydrated Na_xCoO_2

Geometric phases in Josephson-coupled T-breaking superconductors

Recall the phase-sensitive measurements that confirmed *d* order in the cuprates: Tunneling into two faces of a single crystal gives a **positive** sign for *s* order, but a **negative** sign for *d* order.





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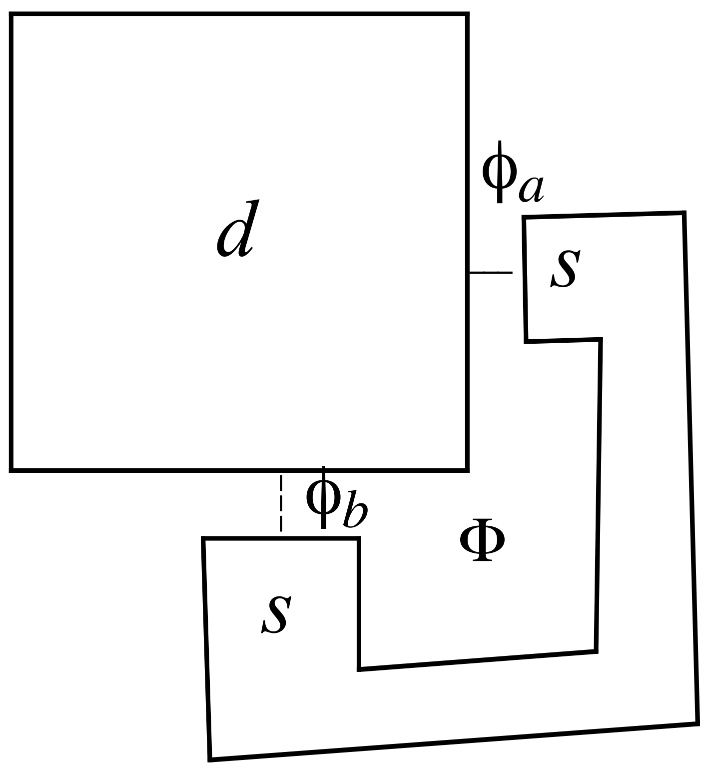
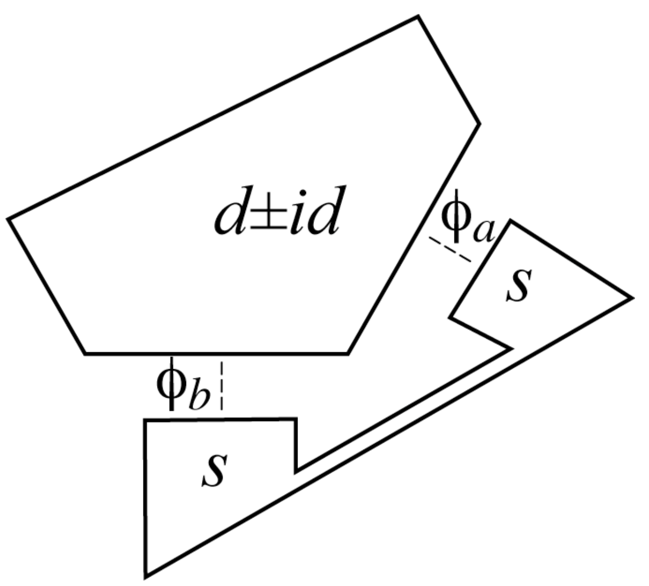


A real superconductor like *p* or *d* generates only phases 0 or π from such angle-resolved tunneling.

Complex superconductors like *p+ip* and *d+id* generate continuously variable phases; in turning by a geometrical angle, a Cooper pair picks up a phase.



Josephson-junction arrays: classical treatment
Statistical physics of geometric phases in tunneling

We describe each superconducting grain by a superconducting phase and an Ising variable: the Ising variable s=±1 describes whether the grain has order parameter p_x+ip_y or p_x-ip_y.

The (classical) energy depends on the gauge-invariant phase difference across the junction, which includes both variables.

$$H = -E_J \sum_{\langle ij \rangle} \cos \theta_{ij}$$

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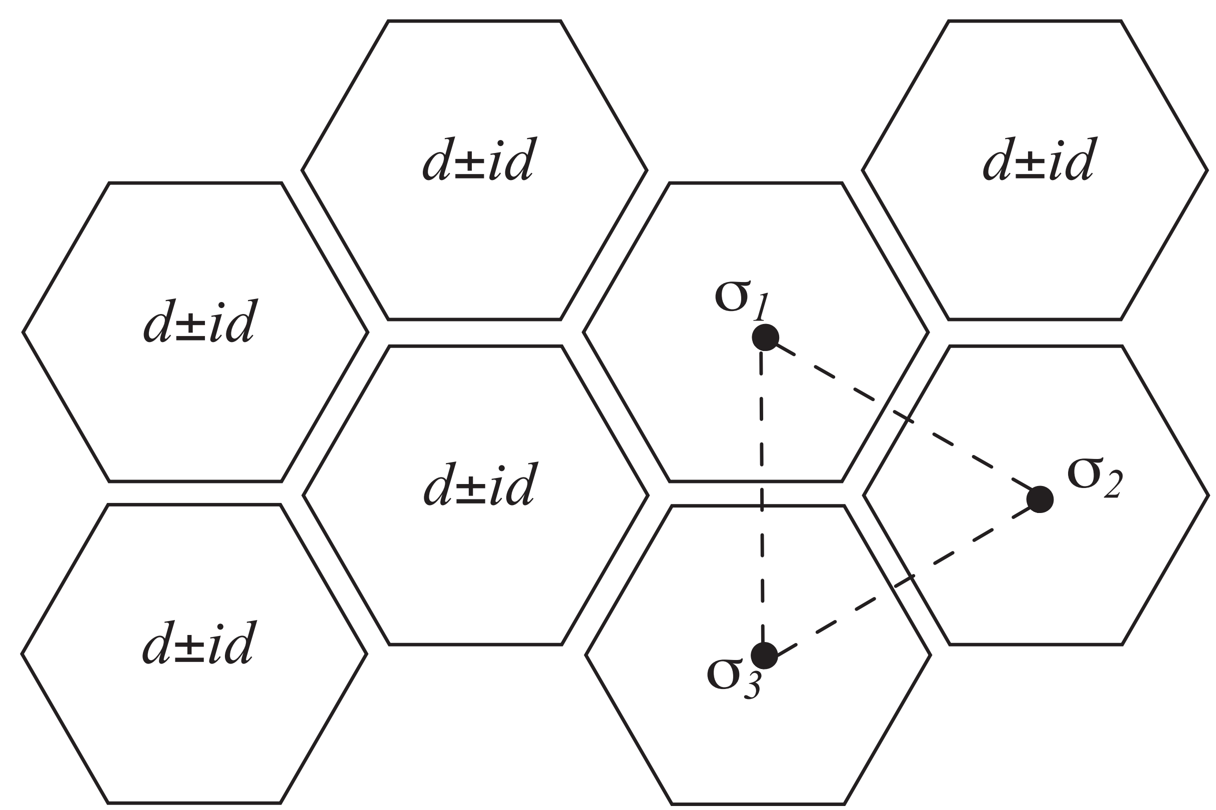
Questions:

How does this geometric coupling affect superconductivity?

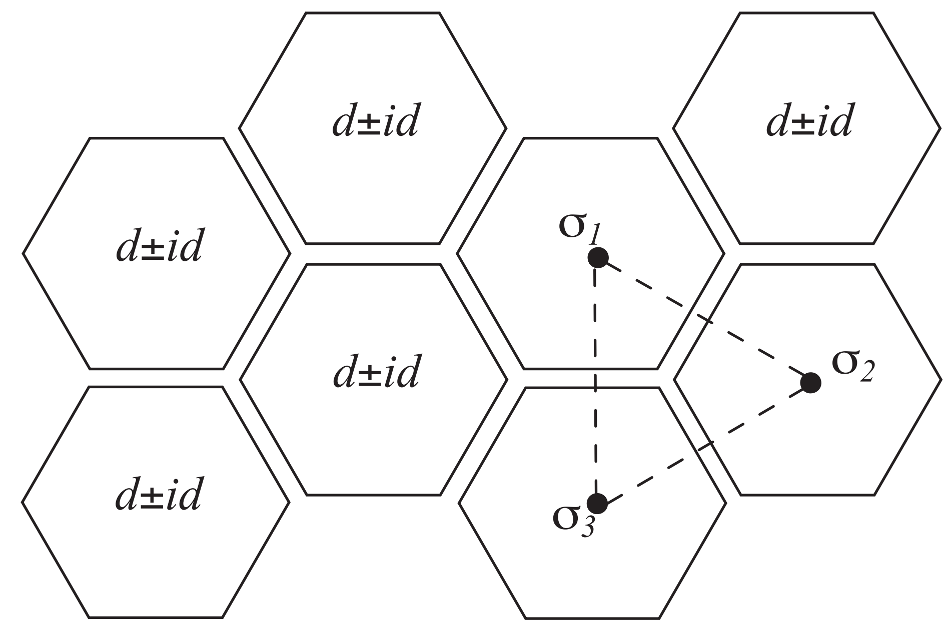
When is there macroscopic **T**-breaking? (When is there Ising order)

Example: hexagonal lattice



Josephson-junction arrays: classical treatment

Around each triangular face of the lattice, the Ising variables generate a gauge flux for the superconducting phases.



Single-valuedness requires that the phase on moving a Cooper pair around a plaquette be equivalent to 0:

$$\frac{2k\pi}{3}(s_1 + s_2 + s_3) + \theta_{12} + \theta_{23} + \theta_{31} = 2\pi n.$$

$$k = 1 \text{ for } p \pm ip,\ \ k = 2 \text{ for } d \pm id$$

$$H = -E_J \sum_{\langle ij \rangle} \cos\theta_{ij}$$

Hence for this lattice, the only ground states are uniform in **both** the Ising and phase variables.
(as Landau-Ginzburg theory predicts)

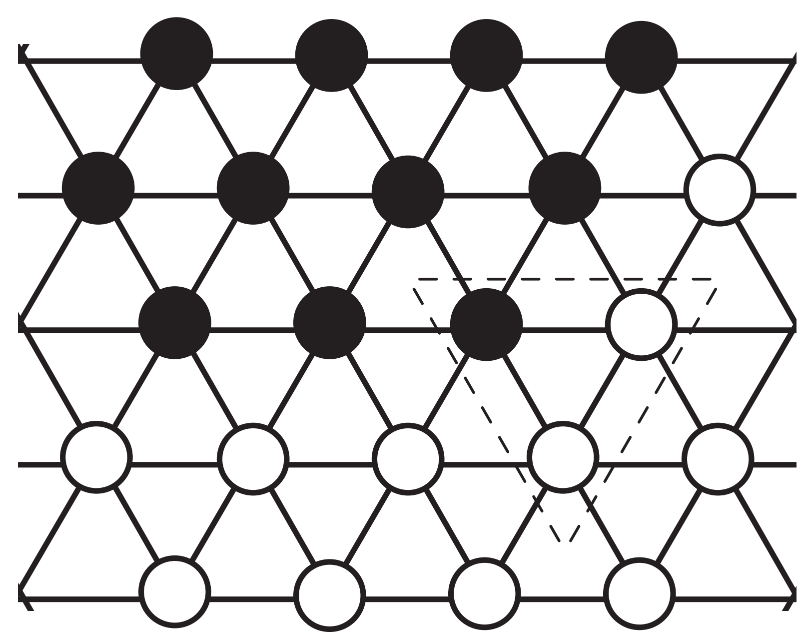
Josephson-junction arrays: classical treatment
Introduction to defects (fractional vortices)

On a triangular face, if all three corners are up (s=1) then there is no frustration of the superconducting phase, and similarly if all three corners are down.

What if two are up and one is down, or vice versa? Such a configuration looks like a fractional vortex from the point of view of the superconducting phases.

Since vortices in 2D are known to mediate the Kosterlitz-Thouless phase transition, one might wonder whether these fractional vortices are important.

For this lattice, fractional vortices are **confined**: only domain wall kinks carry an unscreened fractional vortex. This is not be the case for other lattices.



Josephson-junction arrays: classical treatment
General approach for all lattices

Separate free energy into XY (with fluxes) and entropic parts

$$
\begin{aligned}
Z &= \sum_{s_i,\phi_i} e^{-\beta H} \\
&= \sum_{\phi_i, F^{\langle ijk\rangle}} Z_e(F^{\langle ijk\rangle}) e^{-\beta H(\phi_i, F^{\langle ijk\rangle})}, \\
Z_e(F^{\langle ijk\rangle}) &= \sum_{s_i} \delta(F^{\langle ijk\rangle} - 2\pi(s_i + s_j + s_k)/3). \qquad (1)
\end{aligned}
$$

The entropic part is simply geometry: how many configurations of Ising variables s correspond to a given arrangement of fluxes F ?

For the triangular lattice there are no free fractional fluxes (plaquettes with nontrivial geometric phase), because the **entropic** term is **confining**.

Josephson-junction arrays: classical treatment

Now consider the square lattice:

$$\frac{k\pi}{2}(s_1 + s_2 + s_3 + s_4) + \theta_{12} + \theta_{23} + \theta_{34} + \theta_{41} = 2\pi n$$

For k=2 (d±id) all Ising configurations are U(1) gauge equivalent to the uniform configuration: there is no coupling between the Ising variables and superconducting variables.
The Ising variables are disordered even at T=0.

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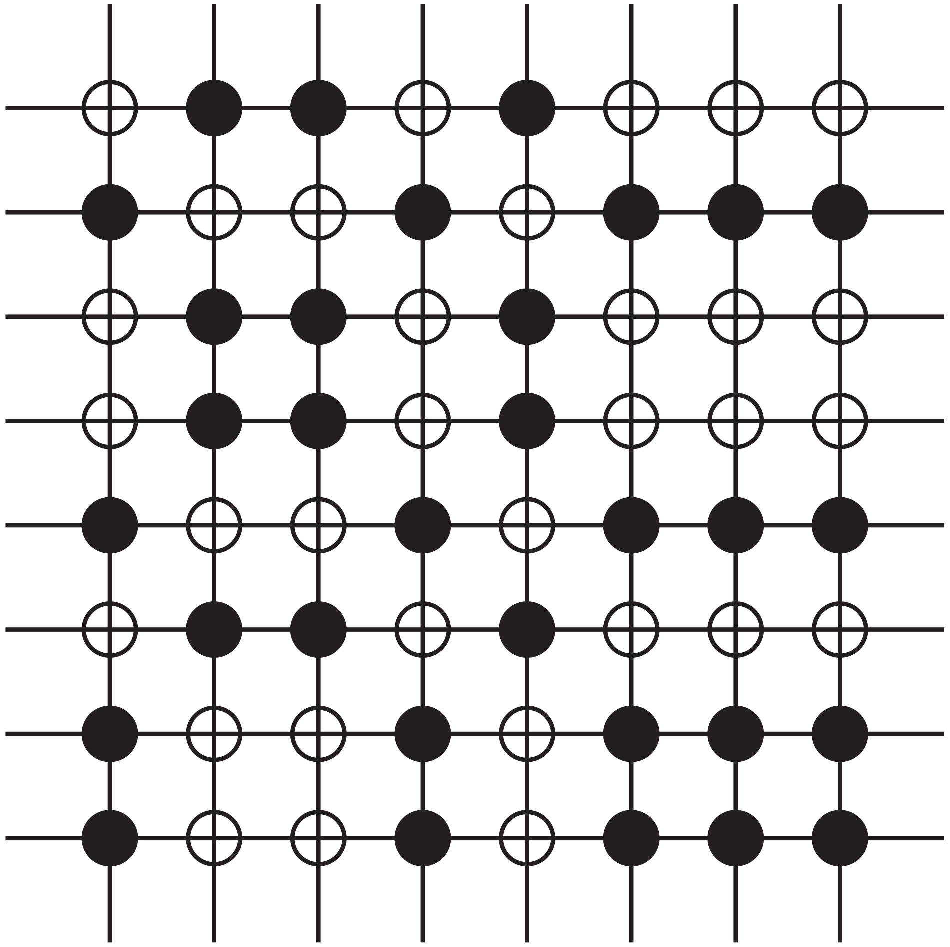
For k=1 (p±ip) ground states have 0,2, or 4 up-spins on each plaquette. This is a “bond-ordered” state with one-dimensional entropy: choosing the Ising spins on one row and one column determines all others.

Such one-dimensional families of ground states also appear in certain 2D frustrated magnets (more later).

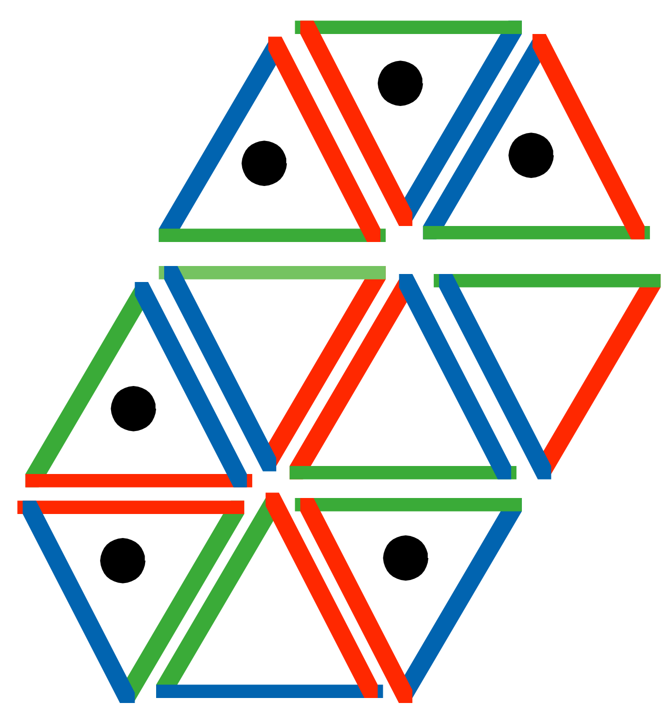


Josephson-junction arrays: classical treatment

Most interesting of all is the honeycomb lattice: the ground-state constraint

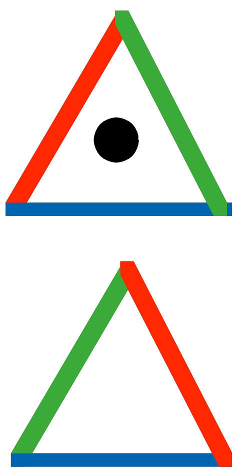
$$\frac{k\pi}{3}(s_1 + s_2 + s_3 + s_4 + s_5 + s_6) = 2\pi n, \quad n \in \mathbf{Z}.$$

can be mapped onto a dual version of Baxter's "three color model": the colorings of the **bonds** of the honeycomb lattice by 3 colors, in such a way that all three colors meet at each vertex.



The connection is that the Ising variables give the **chirality** of the three colors meeting at a vertex.

There is an extensive ground-state entropy with **critical** Ising correlators (confirmed numerically). Melting occurs by fractional vortices (cf. kagomé XY AF).



Defects of Ising variables seem to realize orbifolds of 2D c=2 *SU(3)* CFT.

Josephson-junction arrays: summary of classical results

So far we have seen:
perfect T-breaking order (hexagonal lattice)
disorder or **bond order** (square lattice, depending on *d* or *p* case)
criticality and **melting by fractional vortices** (honeycomb lattice)

For real experiments on random polycrystalline samples, it thus seems likely that the global **T**-breaking transition is suppressed significantly below the superconducting transition, perhaps all the way to zero temperature.
The geometric dependence of Josephson tunneling is crucial.

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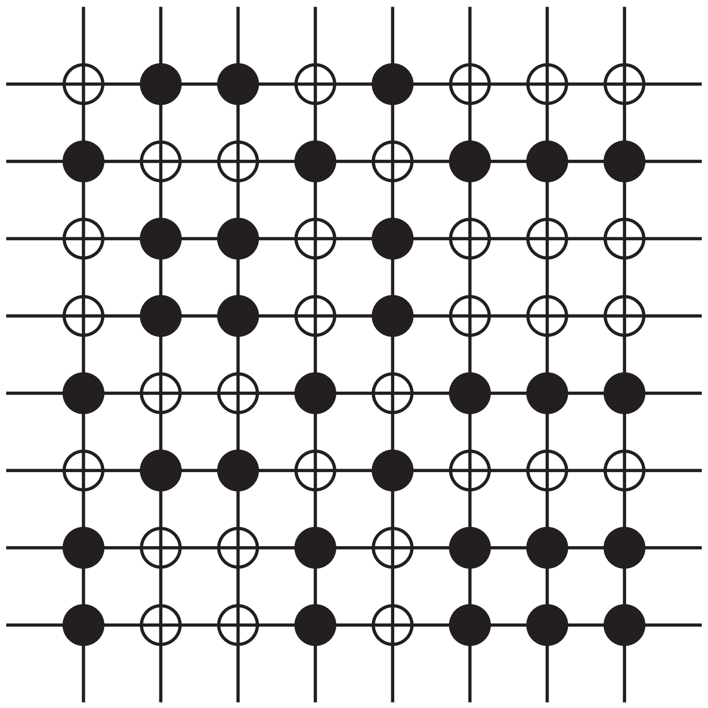
The geometric dependence of Josephson tunneling is crucial.

Consider again the case of p±ip superconductors on the square lattice, where we found a **one-dimensional** set of ground states with **bond order.**

These ground-state phenomena are familiar from some frustrated magnets in 2D...



Dimensional reduction in classical case

The configurations satisfying the square-lattice constraint of 0, 2, or 4 up-spins per plaquette are also the ground states of the gauge-like Hamiltonian

$$H = -K \sum_{\square} s^1_{\square} s^2_{\square} s^3_{\square} s^4_{\square}, \quad s = \pm 1$$

We will use this as an approximate description of the finite-energy physics: it corresponds to “integrating out” the superconducting phases, consistent with symmetry. The “gauge group” is one-dimensional and consists of flipping all spins along one row or one column.

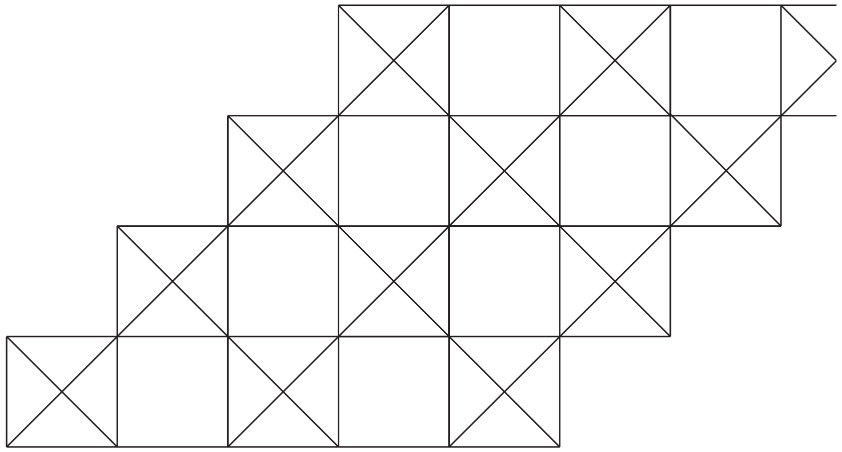
The main approximation (appropriate at high temperatures) is that the vortex-vortex interaction is short-ranged.

There are other physical situations where similar “right-angle ice” models appear:

Frustrated magnetism on the checkerboard lattice

Now the JJ array looks very much like the *1/S* theory of the Heisenberg antiferromagnet on the checkerboard lattice:
The “tetrahedra” with diagonal interactions are intended to model the physics of corner-sharing tetrahedra in 3D pyrochlores.

In the classical limit (infinite S), there is a highly degenerate set of ground states, labeled by Ising variables.



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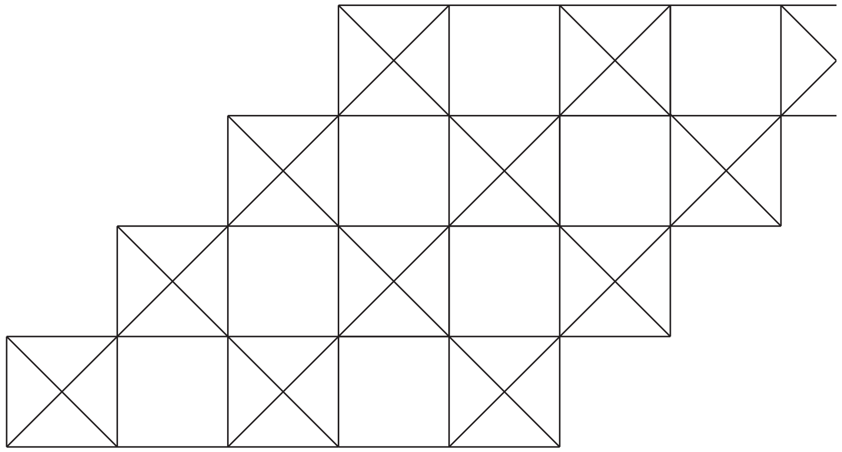

In the classical limit (infinite S), there is a highly degenerate set of ground states, labeled by Ising variables.

Turning on 1/S interactions generates an effective Hamiltonian within this ground-state subspace (Henley, Tchernyshyov et al.).

The effective Hamiltonian is **exactly** the same as before, but with an additional constraint: on the “tetrahedra”, the total spin must be exactly 0, rather than 0, -4, or +4 as before.

Point: Real materials can have symmetries that generate 4-spin interactions without 2-spin interactions.

Solution of a 2D Ising model

$$H = -K \sum_{\square} s^1_{\square} s^2_{\square} s^3_{\square} s^4_{\square}, \quad s = \pm 1$$

Recall the following quick way to solve the **one-dimensional** Ising model in zero field: each bond can be chosen to be independently ferromagnetic or antiferromagnetic, so

$$\frac{F}{N} = \frac{-T \log Z}{N} = -T \log(2 \cosh J/T)$$

Solution of a (trivial) 2D Ising model

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Recall the following quick way to solve the one-dimensional Ising model in zero field: each bond can be chosen to be independently ferromagnetic or antiferromagnetic, so

$$\frac{F}{N} = \frac{-T \log Z}{N} = -T \log(2 \cosh J/T)$$

The above **2D** model is exactly the same: each **face** can be chosen to be independently frustrated or unfrustrated, and the free energy per site is the same as for the **1D** Ising model.

There is no phase transition **at any temperature**, so this 2D Ising model is like a generic 1D model (trivial “dimensional reduction”).

For the checkerboard lattice problem, we find that the constraint on tetrahedra generates a phase transition at finite temperature: dimensional reduction only appears in ground-state properties.

Quantum fluctuations in 2D

The previous calculation shows that the ordered state is unstable to thermal fluctuations.

What about quantum fluctuations at T=0? Does tunneling of the order parameter cause a quantum phase transition?

$$H = -K \sum_{\Box} s_1^z s_2^z s_3^z s_4^z - h \sum_i s_i^x$$

It turns out that this model bears a deep connection to the quantum Ising chain, a **one-dimensional** quantum model: among other similarities, it has essentially the same strong-weak coupling duality.

Quantum fluctuations in 2D

In one quantum or two classical dimensions, many nonperturbative methods apply, such as the exact coupling duality (Kramers-Wannier) of the 2D Ising model.

In terms of the quantum Ising model, this duality interchanges the *K* and *h* terms: hence the critical point must lie at *K*=*h*.

A related duality interchanges K and h terms in our model:

$$H = -K \sum_{\square} s_1^z s_2^z s_3^z s_4^z - h \sum_i s_i^x$$

Construction of dual operators:
(here *j*<*i* means a string in 1D or quadrant in 2D)

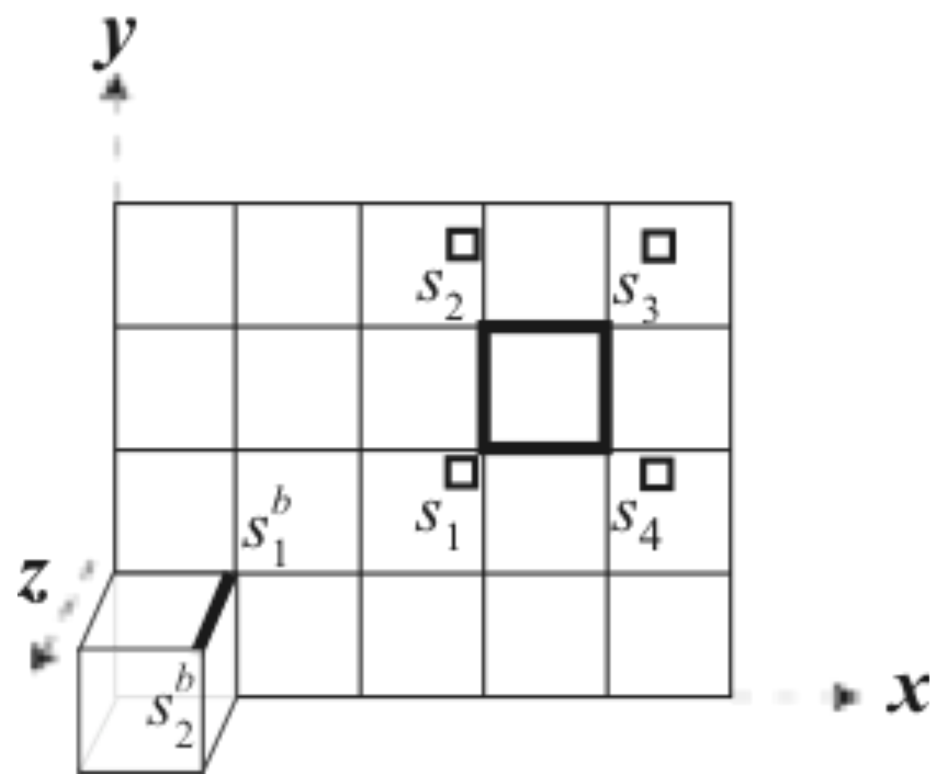
$$\sigma_i^x = s_1^z s_2^z s_3^z s_4^z, \quad \sigma_i^z = \prod_{j<i} s_j^x$$

Quantum fluctuations in 2D

$$H = -K \sum_{\square} s_1^z s_2^z s_3^z s_4^z - h \sum_i s_i^x$$

Why does this model have a self-duality?

Heuristic argument: the classical analogue is a hybrid of "gauge" and "Ising" interactions in 3D.



Recall that the 3D Ising model is dual to an Ising gauge theory; this hybrid theory is self-dual, and the duality interchanges the Ising and gauge terms (cf. Savit et al., late 1980s).

Critical theory and analogy to sliding Luttinger liquid

Since we have a continuous phase transition, it should be described by a continuum theory. Our starting lattice model had the full symmetry of the square lattice.

Normally this symmetry is promoted to full rotational symmetry at the critical point, but here it is not. Even though the original lattice problem has nonvanishing 2D correlators, the critical theory is (we believe) a set of decoupled 1D theories. Couplings of these theories are “irrelevant” but dominate the 2D correlations.

A similar conclusion is reached for quantum Hall stripe phases by Lawler and Fradkin: an initially 2D, but anisotropic, hydrodynamic theory flows toward decoupled 1D theories (a “sliding Luttinger liquid” phase).

The *U(1)* limit of our Z2 theory underlies the “Bose metal” of Paramekanti, Balents, Fisher (2003), with two “sliding” symmetries. We also found a phase diagram for self-dual theories with *N*-fold symmetry breaking.

Conservation laws and connection to Kitaev model

$$H = -K \sum_{\square} s_1^z s_2^z s_3^z s_4^z - h \sum_i s_i^x$$

This model has an infinite, but not extensive, number of conserved quantities: the product of s^x along any row, or any column, commutes with the Hamiltonian.

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Consider the "Kitaev model" for bond variables on the square lattice:

$$H = -K_1 \sum_{\Box} \sigma_1^z \sigma_2^z \sigma_3^z \sigma_4^z - K_2 \sum_{+} \sigma_1^x \sigma_2^x \sigma_3^x \sigma_4^x.$$

Now all the terms in the Hamiltonian commute: there is a 2D set of conserved quantities, and the problem is fully solvable.

Conclusions and references

Several physical problems (including **frustrated magnetism** and geometric dependence of Josephson tunneling) naturally generate many-spin effective interactions.

Precise dimensional reduction from 2D to 1D in quantum models seems to result from having a 1D-infinite set of conserved quantities because of the many-spin interaction. Similar models can be made in d>2 by increasing the number of spins in the interaction.

In our models, duality gives a result for how the classical phase is modified by quantum fluctuations. An active area: what happens to other classical states, like the “photons” of the 3D pyrochlore?

References:

Classical superconducting arrays: JEM and D.-H. Lee, PRB 2004.
Quantum arrays and dimensional reduction:
C. Xu and JEM, PRL 2004 and cond-mat.