~~Exotic Orderings in~~

Two-Leg Spin Ladder with Four-spin exchanges

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Ring - Exchange

→ Importance of 4-Spin Cyclic Exchange ⇝

* Magnetism of ^3He adsorbed on Graphite surfaces Roger et al. (83)

* Spin Structure of a Wigner Crystal Chakraverty et al. (99)

* Magnetic Properties of
  La_2CuO_4 Coldea et al. 00
  $La_xCa_{14-x}Cu_{24}O_{41}$ Brehmer et al. (9

→ Route to Exotic Spin Liquid Phases in $D \geq 2$:

Balents, Fisher, Girvin, Lhuillier, Misguich, Paramekanti ...

* Topological Order

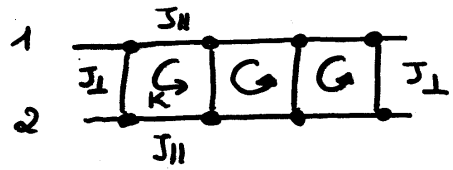
* Fractionalization

* Bose Metal phase

⋮

Scalar Chirality Phase

Two-Leg Spin Ladder with a Ring exchange:



$\vec{S}_{1,2,n}$ spin-½ operators

$$\mathcal{H}_{\square} = J_{\parallel} \sum_n \left\{ \vec{S}_{1,n} \cdot \vec{S}_{1,n+1} + \vec{S}_{2,n} \cdot \vec{S}_{2,n+1} \right\}$$
$$+ J_{\perp} \sum_n \vec{S}_{1,n} \cdot \vec{S}_{1,n+1} + K \sum_n \left[P_{\square}(n,n+1) + P_{\square}^{-1}(n,n+1) \right]$$

Discovery: Läuchli, Schmid, Troyer (2003)
Hikihara, Momoi, Hu (2003)
Momoi, Hikihara, Nakamura, Hu (2003)

Scalar Chirality Phase

$$\langle \vec{S}_{1,n} \cdot (\vec{S}_{1,n+1} \wedge \vec{S}_{2,n+1}) \rangle \neq 0$$

Wen, Wilczek, Zee (89)
Wiegman (89)

Spontaneous Breaking of the Time-Reversal Symmetry !!

Spin - Chirality Duality

Hikihara et al. (2003)
Momoi et al. (2003)

*Exercise!!

$$\vec{S}_1, \vec{S}_2 \quad \text{spin-}\tfrac{1}{2} \xrightarrow{D} \begin{cases} \vec{S} = \frac{1}{2}(\vec{S}_1+\vec{S}_2) - \vec{S}_1\wedge\vec{S}_2 \\ \vec{T} = \frac{1}{2}(\vec{S}_1+\vec{S}_2) + \vec{S}_1\wedge\vec{S}_2 \end{cases}$$

$\vec{S}, \vec{T}$ again spin-1/2 operators!

* Spin-Chirality Duality:

$$\begin{cases} \vec{S}_{1,n} - \vec{S}_{2,n} \xrightarrow{D} 2\,\vec{S}_n \wedge \vec{T}_n \\ -2\,\vec{S}_{1,n}\wedge\vec{S}_{2,n} \xrightarrow{D} \vec{S}_n - \vec{T}_n \end{cases}$$

$$O_{SD} = (-1)^n\left[\vec{S}_{1,n}\cdot\vec{S}_{1,n+1} - \vec{S}_{2,n}\cdot\vec{S}_{2,n+1}\right] \xleftrightarrow{D} O_{SC} = (-1)^n\left[(\vec{S}_n+\vec{T}_n)\cdot(\vec{S}_{n+1}\wedge\right. + n \leftrightarrow n+1$$

staggered Dimerization

staggered Scalar chiral Order

* Exact Scalar chiral GS:

- Generalized 2-leg ladders with exact staggered dimerization ground states
  Kolezhuk, Mikeska

$\downarrow$ Spin-Chirality Duality

- Models with Exact Scalar chiral

Two-leg Ladder with 4-Spin [redacted]

$$\mathcal{H} = J_1 \sum_n \left\{ \vec{S}_{1,n} \cdot \vec{S}_{1,n+1} + 1 \to 2 \right\} + J_\perp \sum_n \vec{S}_{1,n} \cdot \vec{S}_{2,n}$$
$$+ J_d \sum_n \left\{ \vec{S}_{1,n} \cdot \vec{S}_{2,n+1} + \vec{S}_{1,n+1} \cdot \vec{S}_{2,n} \right\}$$
$$+ J_{rr} \sum_n (\vec{S}_{1,n} \cdot \vec{S}_{2,n})(\vec{S}_{1,n+1} \cdot \vec{S}_{2,n+1})$$
$$+ J_{ll} \sum_n (\vec{S}_{1,n} \cdot \vec{S}_{1,n+1})(\vec{S}_{2,n} \cdot \vec{S}_{2,n+1})$$
$$+ J_{dd} \sum_n (\vec{S}_{1,n} \cdot \vec{S}_{2,n+1})(\vec{S}_{1,n+1} \cdot \vec{S}_{2,n})$$

Most General Translation-invariant 2-leg Ladd

- SU(2) interactions 2 neighboring rungs
- $\mathbb{Z}_2$ $1 \leftrightarrow 2$ symmetry

<u>Questions</u>: Exotic $T=0$ Phases ?

Quantum Phase transitions ?

<u>Approach</u>: SU(4) Symmetry

Maximal symmetry for 2 spins 1/2

Wigner 1937

Low Energy effective Field Theory

→ <u>SU(4) point:</u> $J_{\parallel} = 4J_1$, $J_{\perp} = J_d = J_{rr} = J_{dd} = C$

$$\mathcal{H}_{SU(4)} = J_1 \sum_n \sum_{A=1}^{15} S_n^A S_{n+1}^A \qquad S_n^A \text{ SU(4) spin}$$

Sutherland Model Integrable

↳ $SU(4)_1$ QCP with $c = 3$ Affleck 89

→ <u>Conformal embedding:</u> $SU(4)_1 \sim SU(2)_2 \times SU(2)$

⇝ 2 Triplets of Majorana (Real) Fermio

$\vec{\xi}_{R,L}$ and $\vec{\chi}_{R,L}$

→ <u>Most General effective FT:</u>

$$SU(4) \longrightarrow SU(2) \times \mathbb{Z}_2$$

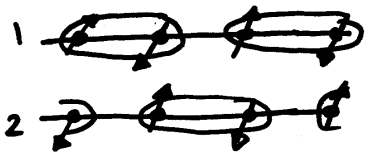
$$\mathcal{H}_{eff} = \mathcal{H}_0 + (g_1+g_2)\left[(\vec{\xi}_R\cdot\vec{\xi}_L)^2 + (\vec{\chi}_R\cdot\vec{\chi}_L)^2\right]$$
$$+ (g_1-g_2)\left[(\vec{\xi}_R\cdot\vec{\chi}_L)^2 + (\vec{\chi}_R\cdot\vec{\xi}_L)^2\right]$$
$$+2(g_3+g_4)(\vec{\xi}_R\cdot\vec{\xi}_L)(\vec{\chi}_R\cdot\vec{\chi}_L) + 2(-g_3+g_4)(\vec{\xi}_R\cdot\vec{\chi}_L)(\vec{\chi}_R\cdot\vec{\xi}$$
$$- \frac{g_5}{2}(\vec{\xi}_R\cdot\vec{\chi}_R)(\vec{\xi}_L\cdot\vec{\chi}_L) - \frac{i g_6}{2}(\vec{\xi}_R\cdot\vec{\chi}_R + \vec{\xi}_L\cdot\vec{\chi}_L)$$

⇝ 1-Loop RG

Strong Coupling Phases

$g_i(t) \sim \frac{r_i}{t_0 - t}$ Lin et al. (98)

→ Staggered Dimerization:



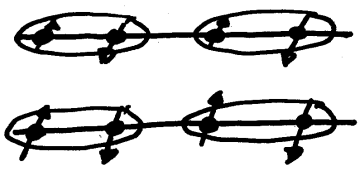
$O_{SD} \sim i\left[\vec{\xi}_R \cdot \vec{\xi}_L - \vec{\chi}_R \cdot \vec{\chi}_L\right]$

→ Scalar chiral Order:

$O_{SC} \sim i\left[\vec{\xi}_R \cdot \vec{\chi}_L + \vec{\chi}_R \cdot \vec{\xi}_L\right]$

$\langle O_{SC} \rangle = \pm \Delta_0$

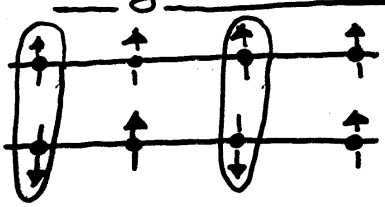
→ Columnar Dimerization:



$O_D^{lat} = (-1)^n \left[\vec{S}_{1,n} \cdot \vec{S}_{1,n+1} + 1 \to 2\right]$

$O_D \sim i\left[\vec{\xi}_R \cdot \vec{\xi}_L + \vec{\chi}_R \cdot \vec{\chi}_L\right]$

→ Rung Dimerization:



$O_{RD}^{lat} = (-1)^n \vec{S}_{1,n} \cdot \vec{S}_{2,n}$

$O_{RD} \sim i\left[\vec{\xi}_R \cdot \vec{\chi}_L - \vec{\chi}_R \cdot \vec{\xi}_L\right]$

→ All these phases are described in the IR:

$$\mathcal{H}_{GN} = \mathcal{H}_0 + g^* \left(\sum_{A=1}^{6} \eta_R^A \eta_L^A \right)^2$$

SO(6) Gross-Neveu model

Dualities

→ **SO(2) Rotations:** $\mathcal{R}_r(\theta)$ $r = R, L$

$$\begin{pmatrix} \xi_r^a \\ \chi_r^a \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \xi_r^a \\ \chi_r^a \end{pmatrix}$$

$\mathcal{H}_0$ Invariant

→ **Spin-chirality Rotation:** $\mathcal{U}_\theta = \mathcal{R}_R(\theta) \times \mathcal{R}_L(\theta)$

$$\begin{pmatrix} O_{SD} \\ O_{SC} \end{pmatrix} \xrightarrow{\mathcal{U}_\theta} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} O_{SD} \\ O_{SC} \end{pmatrix}$$

$$O_D \xrightarrow{\mathcal{U}_\theta} O_D \,, \quad O_{RD} \xrightarrow{\mathcal{U}_\theta} O_{RD}$$

⇒ $\mathbb{Z}_2$ Duality : $D = \mathcal{U}_{\theta=\pi/2}$

$$\begin{cases} \langle O_{SD} \rangle \xleftrightarrow{D} \langle O_{SC} \rangle \\ O_D \,, O_{RD} \text{ self-dual} \end{cases}$$

Continuum Representation of the spin-chirality Duality
Momoi et al. (03

$$\mathcal{U}_\theta^{\text{Lat}} = e^{i\theta \sum_n [\vec{S}_{1,n} \cdot \vec{S}_{2,n} - 1/4]}$$

→ **Second Duality:** $\tilde{\mathcal{U}}_\theta = \mathcal{R}_R(\theta) \times \mathcal{R}_L(-\theta)$

$$\begin{cases} (O_D, O_{RD}) \text{ Transforms as a doublet} \\ (O_{SD}, O_{SC}) \text{ Invariant} \end{cases}$$

⇒ $\mathbb{Z}_2$ Duality : $\tilde{D} = \tilde{\mathcal{U}}_{\theta=\pi/2}$

$$\begin{cases} \langle O_D \rangle \xleftrightarrow{\tilde{D}} \langle O_{RD} \rangle \\ \text{[illegible]} \quad \text{Self-dual} \end{cases}$$

Self-Dual Manifolds

$$g_2 = \pm g_4$$

→ Invariant under $\mathcal{U}_\theta \nleftrightarrow \widetilde{\mathcal{U}}_\theta \quad \forall \theta$

→ <u>2 symmetric rays in the Infrared (IR)</u>:

$$\begin{cases} (\mathrm{I}): & g_1^* = -g_3^* = g^*, \quad g_6^* = \frac{8}{3} g^*, \quad g_2^* = g_4^* = 0 \\ (\mathrm{II}): & g_1^* = g_3^* = \hat{g}^*, \quad g_6^* = -\frac{8}{3} \hat{g}^*, \quad g_2^* = g_4^* = 0 \end{cases}$$

→ <u>Low-Energy effective Field theories</u>:

* <u>Along (I)</u>:

$$\mathcal{H}_{eff}^{int} = g^* \Big\{ \underbrace{\left[\vec{\xi}_R \cdot \vec{\xi}_L - \vec{x}_R \cdot \vec{x}_L\right]^2}_{-O_{SD}^2} + \underbrace{\left[\vec{\xi}_R \cdot \vec{x}_L + \vec{x}_R \cdot \vec{\xi}_L\right]^2}_{-O_{SC}^2} - \frac{2}{3} g^* \left(\vec{\xi}_R \cdot \vec{x}_R + \vec{\xi}_L \cdot \vec{x}_L\right)^2$$

≡ Effective theory for the transition: Staggered Dimerized phase / Scalar chirality Phase

* <u>(II)</u>:

$$\mathcal{H}_{eff}^{int} = g^* \Big\{ \underbrace{\left[\vec{\xi}_R \cdot \vec{\xi}_L + \vec{x}_R \cdot \vec{x}_L\right]^2}_{-O_D^2} + \underbrace{\left[\vec{\xi}_R \cdot \vec{x}_L - \vec{x}_R \cdot \vec{\xi}_L\right]^2}_{-O_{RD}^2} \Big\} + \frac{2}{3} g^* \left(\vec{\xi}_R \cdot \vec{x}_R + \vec{\xi}_L \cdot \vec{x}_L\right)^2$$

Transition: ___ Columnar Dimerization / Rung Dimerization

U(1) Quantum Criticality

- **Naive Symmetry of the Self-dual theory:**

$$U(1) \times SU(2)$$

$$\swarrow$$

$$u_\theta \times \tilde{u}_\theta$$

- **Introduction of Dirac Fermions:**

$a=1,2,3 \qquad \psi_{aR,L} = \frac{1}{\sqrt{2}}\left(\xi^a_{R,L} + i\,\chi^a_{R,L}\right)$

$$\begin{cases} u_\theta : & \psi_{aR,L} \longrightarrow e^{i\theta/2}\,\psi_{aR,L} \quad \text{U(1) symmetry} \\ \tilde{u}_\theta & \begin{pmatrix}\psi_{aR}\\ \psi_{aL}\end{pmatrix} \longrightarrow e^{i\gamma_5\theta/2}\begin{pmatrix}\psi_{aR}\\ \psi_{aL}\end{pmatrix} \quad \text{U(1) chiral Symmetry} \end{cases}$$

- **Order Parameter:**

$$O = O_{SD} + i\,O_{SC} = \sum_{a=1}^{3} \psi^+_{aR}\,\psi^+_{aL} \quad \text{Cooper Pairing}$$

- **"Spin - charge" Separation:** "charge" $\leftrightarrow$ Spin-chirality

$$\mathcal{H}_{eff} = \mathcal{H}_C + \mathcal{H}_S \qquad U(1) \times SU(3)$$

$$\begin{cases} * \ \mathcal{H}_C = \frac{v}{2}\left[(\partial_x\varphi)^2 + (\partial_x\theta)^2\right] & \text{Critical! : Luttinger liquid} \\ \qquad u_\theta \times \tilde{u}_\theta \text{ Invariant} & \\ * \ \mathcal{H}_S^{int} = g^* \sum_{A=1}^{8} J^A_R\,J^A_L & SU(3) \text{ Gross-Neveu} \end{cases}$$

Spectral Gap

Andrei-Lowenstein '80

$\Rightarrow$ U(1) Quantum Criticality

Properties of the Quantum Critical Point

* <u>At the U(1) QCP:</u> $\langle O_{SD}\rangle = \langle O_{SC}\rangle = \langle O_{RD}\rangle = \langle O_{D}\rangle = 0$

Power-Law Behaviour: $\langle O_{\underset{SC}{SD}}(x)\, O_{\underset{SC}{SD}}(0)\rangle \sim x^{-x/\xi}$

whereas $\langle O_{\underset{RD}{D}}(x)\, O_{\underset{RD}{D}}(0)\rangle \sim e^{-x/\xi}$

* <u>Small Departure / Self-dual Manifold:</u>

$$\mathcal{H}_{\epsilon} = \epsilon(\mathcal{H}_2 - \mathcal{H}_4)\quad U_\theta \longrightarrow \mathbb{Z}_2 \quad (\theta = \pi/2)$$

$\hookrightarrow$ Spin Gap in the Spin Sector

$$\mathcal{H}_c = \frac{v}{2}\left[\frac{1}{K_c}(\partial_x\varphi)^2 + K_c(\partial_x\theta)^2\right] - \epsilon\cos[illegible]$$

Strongly Relevant $\rightsquigarrow$ Charge Gap

- $\epsilon > 0$ Staggered Dimerization: $\langle O_{SD}\rangle \neq 0$, $\langle O_{SC}\rangle = 0$
- $\epsilon < 0$ Scalar Chirality Phase: $\langle O_{SC}\rangle \neq 0$, $\langle O_{SD}\rangle = 0$

* <u>QCP "belongs" to the Class of "Deconfined QCP":</u>

Senthil, Balents, Sachdev, Vishwanath, Fisher, Science (2004)

Emerging U(1) Quantum Criticality between Néel and VBS phases in $D=2$.

Conclusion

→ Determination of the $T=0$ Gapped Phases of the Generalized two-leg ~~spin~~ ladder in the vicinity of a SU(4) Point

- Dimerized Phases
- Scalar chiral Phase

→ Quantum Phase transitions:

Staggered Dimerized / Scalar chiral

Columnar Dimerization / Rung dimerization

Second Order Phase transition with U(1) Self-dual Critical Point

→ Phases, QPT of Doped two-leg Ladder with generalized interactions

U(4) symmetry Schulz (98)

or SO(8) symmetry Lin et al. (98)