Complementary colors of colorons: The elementary excitations of the SU(3) Haldane–Shastry model

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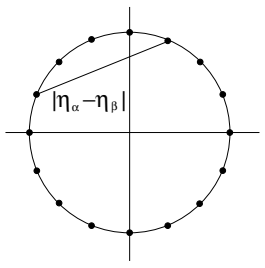
1 Haldane–Shastry model

N sites with $\frac{1}{2}$ spins on the unit circle



$$\eta_\alpha = \exp\left(\frac{2\pi i}{N}\alpha\right),$$

$$\alpha = 1, \ldots, N$$

Hamiltonian:

$$H_{\mathrm{HS}} = \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N} \frac{\vec{S}_\alpha \cdot \vec{S}_\beta}{|\eta_\alpha - \eta_\beta|^2}$$

Symmetry generators:

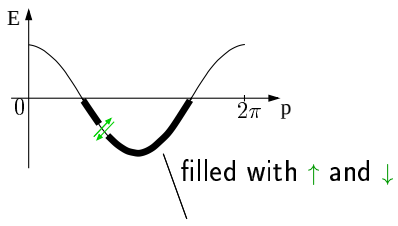
$$\vec{S} = \sum_{\alpha=1}^{N} \vec{S}_\alpha\,, \quad \vec{\Lambda} = \frac{i}{2}\sum_{\alpha\neq\beta}^{N} \frac{\eta_\alpha + \eta_\beta}{\eta_\alpha - \eta_\beta}\left(\vec{S}_\alpha \times \vec{S}_\beta\right)$$

generate the Yangian, an infinite dimensional algebra
⇒ infinite number of conserved quantities
⇒ integrable system

2 Ground state

1D tight-binding:

$$H_{\mathrm{tb}} = -t \sum_{\langle\alpha\beta\rangle} c_\alpha^\dagger c_\beta$$



HS ground state:

$$|\Psi_0\rangle = P_{\mathrm{GW}}\,|\Psi_{\mathrm{SD}}^N\rangle \equiv \prod_{\alpha=1}^{N}(1 - n_{\alpha\uparrow}n_{\alpha\downarrow})\,|\Psi_{\mathrm{SD}}^N\rangle$$

- P_{GW} eliminates doubly occupied sites
- $\left[\vec{S}_\alpha, P_{\mathrm{GW}}\right] = 0$: $|\Psi_{\mathrm{SD}}^N\rangle$ singlet → $|\Psi_0\rangle$ singlet
- $|\Psi_0\rangle$ represents a spin liquid

3 Spinons

N odd:

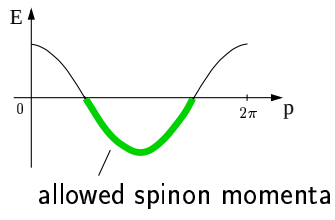
$$|\Psi_\alpha\rangle = P_{\mathrm{GW}}\,c_{\alpha\downarrow}\,|\Psi_{\mathrm{SD}}^{N+1}\rangle = (-1)^\alpha P_{\mathrm{GW}}\,c_{\alpha\uparrow}^\dagger\,|\Psi_{\mathrm{SD}}^{N-1}\rangle$$

creates inhomogeneity in spin and charge

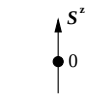
⇒ after projection only inhomogeneity in spin
spin $\frac{1}{2}$, charge 0 → spin-charge separation

energy eigenstates:

$$|\Psi_k\rangle = \sum_{\alpha=1}^{N} e^{\frac{2\pi i}{N}k\alpha}\,|\Psi_\alpha\rangle = P_{\mathrm{GW}}\,c_{k\downarrow}\,|\Psi_{\mathrm{SD}}^{N+1}\rangle$$

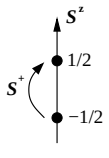


4 Representations of SU(2)



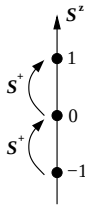
Singlet:

dimension: $n = 1$, e.g. mesons like $\pi^\pm, \pi^0$



Doublet:

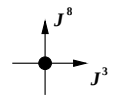
$n = 2$, fundamental representation, e.g. electron



Triplet:

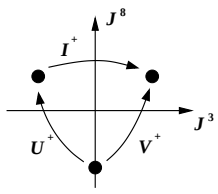
$n = 3$, e.g. $W^\pm, Z^0$

5 Representations of SU(3)



1 representation:

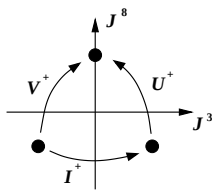
$n = 1$, white representation in QCD, e.g. proton, neutron



3 representation:

$n = 3$

fundamental representation, e.g. quarks in QCD



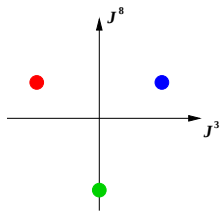
$\bar{3}$ representation:

$n = 3$

complex conjugated to 3, e.g. antiquarks

6 SU(3) Haldane–Shastry model

N sites with one particle per site, which transforms under the representation 3
⇒ color blue, red, or green

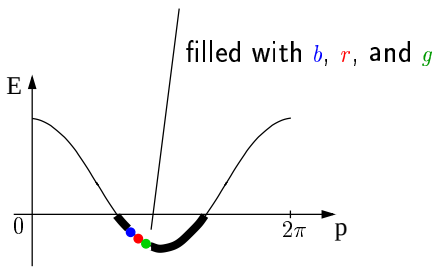


$$H = \left(\frac{2\pi}{N}\right)^2 \sum_{\alpha<\beta}^{N}\sum_{j=1}^{8} \frac{J_\alpha^j J_\beta^j}{|\eta_\alpha - \eta_\beta|^2} \sim \sum_{\alpha<\beta}^{N}\sum_{i,j=b,r,g} \frac{e_\alpha^{ij} e_\beta^{ji}}{|\eta_\alpha - \eta_\beta|^2}$$

with J_α^j the generators of 3

Ground state:

$$|\Psi_0\rangle = P_{\mathrm{GW}}\,|\Psi_{\mathrm{SD}}\rangle$$



ground state is white, i.e. an SU(3) singlet

7 Colorons

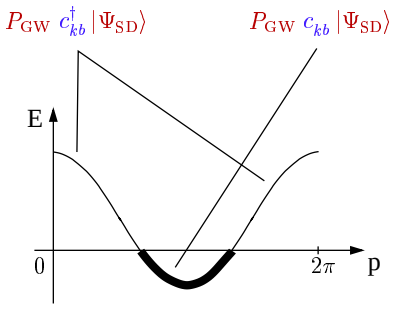
two possibilities:

$$P_{\mathrm{GW}}\,c_{\alpha b}^\dagger\,|\Psi_{\mathrm{SD}}\rangle = P_{\mathrm{GW}}\,c_{\alpha r}c_{\alpha g}\,|\Psi_{\mathrm{SD}}\rangle$$

or

$$P_{\mathrm{GW}}\,c_{\alpha b}\,|\Psi_{\mathrm{SD}}\rangle$$

In momentum space:



They cannot both be elementary excitations!
What are the physically correct excitations?

8 Technique

$P_{\mathrm{GW}}\,c_{kb}\,|\Psi_{\mathrm{SD}}\rangle$ can be written as

$$\Psi_n(z;w) = \frac{1}{N}\sum_{\alpha=1}^{N}(\eta_\alpha^*)^n \prod_{i=1}^{M_1}(\eta_\alpha - z_i)\;\Psi_0(z;w)$$

where $k = \frac{4}{3}\pi - \frac{2\pi}{N}\left(n + \frac{1}{3}\right), 0 \le n \le M_1 = (N-2)/3$

with expressions like ($M_1 \le M_2$)

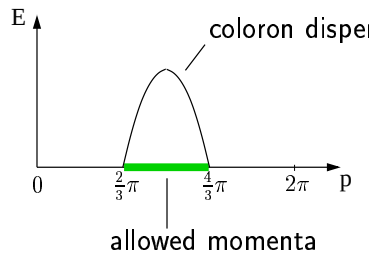
$$\sum_{i=1}^{M_1}\sum_{k=1}^{M_2} \frac{z_i}{z_i - w_k}\left[w_k \prod_{j\neq i}^{M_1}\frac{z_j - w_k}{z_j - z_i}\prod_{l\neq k}^{M_2}\frac{w_l - z_i}{w_l - w_k} - z_i\right] \cdot \sum_{\alpha=1}^{N}\eta_\alpha^{-n}\prod_{j\neq i}^{M_1}(\eta_\alpha - z_j)$$

$$= \left[-\frac{1}{2}n(n+1) + \frac{1}{2}M_1(M_1+1)\right]\sum_{\alpha=1}^{N}\eta_\alpha^{-n}\prod_{i=1}^{M_1}(\eta_\alpha - z_i)$$

one can prove:

$$H\Psi_n(z;w) = E\Psi_n(z;w)$$

⇒ $P_{\mathrm{GW}}\,c_{kb}\,|\Psi_{\mathrm{SD}}\rangle$ are energy eigenstates for $N = 3\kappa - 1$
possess color but no charge → color-charge separation



$$E = E_0 + \frac{3}{4}\left[\frac{\pi^2}{9} - (k-\pi)^2\right]$$

$$k \in \left[\tfrac{2}{3}\pi, \tfrac{4}{3}\pi\right]$$

⇒ $P_{\mathrm{GW}}\,c_{kb}^\dagger\,|\Psi_{\mathrm{SD}}\rangle$ are not energy eigenstates,
they represent two-coloron states

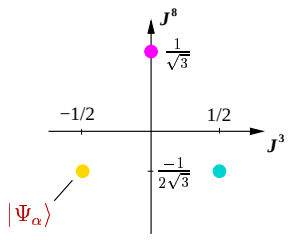
9 Quantum numbers of colorons

$N = 3\kappa - 1$: κ blue, κ red, κ green particles in $|\Psi_{\mathrm{SD}}\rangle$
⇒ in $|\Psi_\alpha\rangle = P_{\mathrm{GW}}\,c_{\alpha b}\,|\Psi_{\mathrm{SD}}\rangle$: $\kappa - 1$ blue,
κ red,
κ green particles

$$J^3\,|\Psi_\alpha\rangle = \frac{1}{2}\sum_{\beta=1}^{N}\left(c_{\beta b}^\dagger c_{\beta b} - c_{\beta r}^\dagger c_{\beta r}\right)|\Psi_\alpha\rangle = -\frac{1}{2}\,|\Psi_\alpha\rangle$$

$$J^8\,|\Psi_\alpha\rangle = -\frac{1}{2\sqrt{3}}\,|\Psi_\alpha\rangle$$

⇒ $|\Psi_\alpha\rangle$ transforms according to $\bar{3}$



at site η_α the color blue is absent
→ complementary color yellow

Statistics: add 3 yellow colorons ⇒ $M_1 \to M_1 - 2$
⇒ statistical parameter for polarized states:

$$g = \frac{2}{3}$$

10 Conclusion

- SU(3) Haldane–Shastry is an integrable model
- $P_{\mathrm{GW}}\,c_{kb}\,|\Psi_{\mathrm{SD}}\rangle$ represent exact coloron states
- colorons possess complementary colors, i.e. they transform under $\bar{3}$
- colorons possess fractional statistics
- result generalizes to SU(n)