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# Dynamical optimization theory of a diversified portfolio

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## Abstract

We propose and study a simple model of dynamical redistribution of capital in a diversified portfolio. We consider a hypothetical situation of a portfolio composed of  $N$  uncorrelated stocks. Each stock price follows a multiplicative random walk with identical drift and dispersion. The rules of our model naturally give rise to power law tails in the distribution of capital fractions invested in different stocks. The exponent of this scale free distribution is calculated in both discrete and continuous time formalism. It is demonstrated that the dynamical redistribution strategy results in a larger typical growth rate of the capital than a static “buy-and-hold” strategy. In the large  $N$  limit the typical growth rate is shown to asymptotically approach that of the expectation value of the stock price. The finite dimensional variant of the model is shown to describe the partition function of directed polymers in random media. © 1998 Elsevier Science B.V. All rights reserved

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## 1. Introduction

The problem of finding an investment strategy with the best long-term growth rate of the capital is of tremendous practical importance. The traditional theory of portfolio optimization is stationary in origin [1]. It answers the question of optimal distribution of the capital between different assets (optimal asset allocation), but in general gives no prescription on how to maintain this optimal allocation at all times. In this work we propose a simple model of *dynamical* allocation of capital. Somewhat counterintuitively, in order to optimize the growth rate of the capital an investor has to sell assets which have increased in price since the last update, and buy those which have decreased. In doing so he sells stocks when they are “overpriced” and buys them when they are “underpriced”, which is clearly advantageous. As we demonstrate below, in our model an investor who actively manages his portfolio in such a fashion almost certainly does better than one who follows a static “buy-and-hold” strategy.

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The nontrivial properties of the problem come from the multiplicative nature of stock price fluctuations. Throughout this manuscript we assume that on timescales of interest to us the prices of individual assets follow a multiplicative random walk. In other words, the ratio of stock prices at two consecutive times, at which the investor buys or sells stock, is a random number, uncorrelated with the current price and with the history of price changes in the past. There are many peculiarities of such noisy multiplicative dynamics, especially regarding expectation values of random variables. Traditional expectation (average) value is of little relevance here. The reason for this is that the dominant contribution to the expectation value of a random variable subject to multiplicative noise comes from exponentially unlikely outcomes when the variable is exponentially large. For any finite number of realizations (and in real world one always deals with just one realization) this expectation (average) value is very unlikely to appear. On the other hand, the *typical* value of such random variable, defined as the median of its probability distribution, constitutes a more realistic property.

Just like in the static portfolio theory, our strategy favors the diversification, i.e. increasing the number of assets in the portfolio. We demonstrate that in our model the diversification reduces fluctuations, and makes the growth rate of the typical value of the capital to be closer to that of its expectation value. However, for any finite number of assets, these two growth rates are still different.

Under the rules of dynamical redistribution of funds, which we employ in this manuscript, the distribution of shares of the total capital invested in individual assets naturally acquires a power law tail. This adds yet another example of how a scale free distribution can arise out of multiplicative dynamics without fine-tuning of any sort. We derive the analytical expression for the exponent  $\tau$  of this power law. Somewhat surprisingly, in the weak coupling limit, corresponding to slow redistribution of funds between the assets, this exponent has a “superuniversal” value  $\tau = 2$ . It gradually increases with the coupling constant and becomes infinite in the limit where the capital is equally redistributed between assets after each time step.

The rules of redistribution of capital can be interpreted as fully connected (infinite-dimensional) limit of the well known statistical model of directed polymers in the presence of quenched disorder. This provides a new and exciting link between the physics of finance, and the problems lying on the forefront of modern theoretical condensed matter physics.

The plan of the manuscript is as follows: to streamline the following introduction of our basic model, in Section 2 we review the well known (and not so well known) properties of a stochastic multiplicative dynamics. We remind the reader the formulas for average and typical value of a single multiplicative random walk, formulate the “continuous” time approach to this problem, and refresh in reader’s memory the formalism of Ito stochastic calculus, necessary for our purposes. Then we review recent results on natural appearance of power law distributions in a situation when a single multiplicative random walk is pushed against lower wall [2,3], preventing the random variable from falling below certain value. Finally, we describe the multiplicative stock price and capital dynamics used throughout this manuscript.

In Section 3 we analyze the behavior of the typical and average values of the capital in a “buy-and-hold” strategy, where the capital was initially equally distributed between  $N$  independent assets with the same typical growth rate and dispersion, and no further redistribution ever took place. We demonstrate that after a logarithmically short initial period of time, the typical growth rate of the capital is limited to the typical growth rate of the price of the assets, and is significantly smaller than their average (expectation) growth rate (or average return per capital of this asset)

In Section 4 we show that the growth rate of investor’s capital can be significantly increased by following an active, dynamic redistribution strategy. In this strategy at each time step the investor sells some shares of every stock with current value of invested capital above the all-stock average, and buys some shares of every stock below this average. We analyze the consequences of this strategy in both discrete and continuous time formalisms and demonstrate that in both cases these rules naturally give rise to a scale free distribution of fractions of individual stock capitals in the total capital. We proceed with deriving analytical expressions for the exponent of this distribution, and the typical growth rate of the capital in this situation. This rate for our strategy proves to be larger than that in the static “buy-and-hold” strategy, which a posteriori justifies our approach. However, as should be expected, the total capital is still subject to the multiplicative noise, and therefore its typical growth rate is still smaller than the average growth rate. We demonstrate that in the limit  $N \rightarrow \infty$  these two rates asymptotically converge as some power of  $1/N$ .

## 2. Review of results for a single multiplicative random walk

### 2.1. Typical and average values of a multiplicative random walk

Consider a stochastic process in which at each time step a variable  $W(t)$  is multiplied by a positive random number  $e^{\eta(t)}$ , where  $\eta(t)$  is drawn from some probability distribution  $\pi(\eta)$ :

$$W(t+1) = e^{\eta(t)} W(t). \quad (1)$$

We adopt the initial condition  $W(0) = 1$ . For the new variable  $h(t) = \ln W(t)$  this process is just a random walk with an average drift  $v = \langle \eta \rangle$  and a dispersion  $D = \langle \eta^2 \rangle - \langle \eta \rangle^2$ . The corresponding equation of motion is simply

$$h(t+1) = h(t) + \eta(t). \quad (2)$$

In recent literature it has been observed that *average* and *typical* values of  $W(t)$  in such a process can be very different. One of the precise definitions of the typical value of a random variable is the *median* [4] of its probability distribution, i.e. for  $W_{typ}$  one has the property that  $Prob(W > W_{typ}) = Prob(W < W_{typ}) = 1/2$ . By definition  $W_{typ}(t) = e^{h_{typ}(t)}$ .

The central limit theorem implies that asymptotically the distribution  $P(h, t)$  can be approximated with a Gaussian

$$P(h, t) = \frac{1}{\sqrt{2\pi Dt}} \exp\left(-\frac{(h - vt)^2}{2Dt}\right). \quad (3)$$

Therefore, the median (as well as average and most probable values) of  $h(t)$  changes linearly with time, and the rate of this change is given by the drift velocity  $v = \langle \eta \rangle$  of the corresponding random walk:  $\ln W_{typ}(t) = \langle \ln W(t) \rangle = \langle \eta \rangle t$ .

On the other hand the expectation (average) value of  $W(t)$  changes as  $\langle W(t+1) \rangle = \langle e^\eta \rangle \langle W(t) \rangle$  (since  $\eta(t)$  and  $W(t)$  are uncorrelated). Hence,  $\ln \langle W(t) \rangle = \ln \langle e^\eta \rangle t$  also depends linearly on time but with a different slope. It is easy to show that for any distribution  $\ln \langle e^\eta \rangle > \langle \eta \rangle$ , so that the average value of  $W$  always grows faster than its typical value and after some time one has  $\langle W(t) \rangle \gg W_{typ}(t)$ . This exponentially large discrepancy between typical and average values of  $W$  is due to the long tails of  $P(W, t)$ , but the events constituting these tails are extremely rare.

For future use we derive analytic expressions for the growth rate of  $\langle W^m(t) \rangle$  in a simple case, when  $\eta$  is drawn from a Gaussian distribution with average value  $v = \langle \eta \rangle$  and dispersion  $D = \langle \eta^2 \rangle - \langle \eta \rangle^2$ . Since the dynamics of  $W^m$  is given by  $W^m(t+1) = e^{\eta(t)m} W^m(t)$ , for  $\langle W^m(t) \rangle$  one has  $\langle W^m(t) \rangle = \langle e^{\eta m} \rangle^t$ . The integral  $\int_{-\infty}^{\infty} d\eta e^{\eta m} e^{-(\eta-v)^2/2D} / \sqrt{2\pi D}$  can be easily taken and is equal to  $e^{m(v+Dm/2)}$ . Therefore, for a Gaussian distribution one has

$$\langle e^{\eta m} \rangle^{1/m} = e^{v+Dm/2}, \quad (4)$$

$$\langle W^m(t) \rangle = e^{m(v+Dm/2)t}. \quad (5)$$

It is important to mention that, although by the virtue of the Central Limit Theorem, for any  $\pi(\eta)$  with a given average  $v$  and dispersion  $D$  the distribution  $P(h, t)$  can be approximated by a Gaussian (3), the precision of this approximation is not sufficient to calculate averages of the type  $\langle W^m(t) \rangle = \int e^{mh(t)} P(h, t) dh$ . This integral is too sensitive to the precise shape of the distribution at the upper tail (or lower tail for  $m < 0$ ). Indeed, the growth rate of  $\ln \langle W^m(t) \rangle$  equal to  $\ln \int_{-\infty}^{\infty} d\eta \pi(\eta) e^{\eta m}$ , depends on the whole shape of  $\pi(\eta)$  and not only on its first and second moments  $v$  and  $D$ .

## 2.2. Multiplicative random walk in the continuous time approach

The above multiplicative process is defined without ambiguity for discrete time. Straightforwardly taking the continuum limit causes problems. It might be useful to rewrite the equation of motion of a multiplicative random walk as a Stochastic Differential Equation (SDE) in continuous time. One should always keep in mind that a stochastic differential equation is nothing more than a convenient notation to describe a stochastic process in discrete time. At the  $n$ th time step of discretized dynamics we define a new “continuous” time variable  $t$  as  $t = n \Delta t$ . Here we introduced a rescaling factor  $\Delta t \ll 1$ , which makes one step of underlying discrete dynamics an

“infinitesimally small” increment of the continuous time  $t$ . In the SDE approach one is limited to Gaussian distributed random variables, so we select a Gaussian distribution of  $\eta(t)$  in our discrete dynamics. Since we want to approximate  $W(t)$  with a continuous function, the difference  $W(t + \Delta t) - W(t)$  after one step of discrete dynamics should be “infinitesimally” small. Therefore, we should select both the average value and the dispersion of the Gaussian variable  $\eta(t)$  to scale as some power of  $\Delta t$ . It turns out to be the right choice to make them both scale linearly with  $\Delta t$ :  $\eta(t) = v\Delta t + \delta\eta(t)$ , where  $\langle \delta\eta(t)^2 \rangle = D\Delta t$ . Now one can write the discrete equation of motion for  $W(t)$  as  $W(t + \Delta t) = e^{v\Delta t + \delta\eta(t)} W(t) \simeq [1 + v\Delta t + \delta\eta(t) + (v\Delta t + \delta\eta(t))^2/2]W(t) \simeq W(t) + ((v + D/2)\Delta t + \delta\eta(t))W(t)$ , where we have dropped all terms smaller than linear in  $\Delta t \ll 1$ . The SDE for  $W(t)$  can now be written as

$$\frac{dW(t)}{dt} = \left( v + \frac{D}{2} \right) W(t) + \tilde{\eta}(t)W(t). \tag{6}$$

Here  $\tilde{\eta}(t) = \delta\eta(t)/\Delta t$  is a usual gaussian “continuous noise” with zero mean and correlations given by  $\langle \tilde{\eta}(t)\tilde{\eta}(t') \rangle = D\delta(t - t')$ . We also assume the absence of correlations between  $W(t)$  and  $\tilde{\eta}(t)$ . This assumption corresponds to selecting the Ito calculus over Stratonovich calculus. Both are just two formal ways of linking the polemic continuum limit and the well defined discrete version.

The nontrivial part of this equation is an extra  $D/2$  term added to a deterministic growth rate of  $W(t)$ . This term is not an artifact of our approach, but has a real physical meaning. Indeed, Eq. (6) can be solved for  $\langle W(t) \rangle$  to give  $\langle W(t) \rangle = \langle W(0) \rangle e^{(v+D/2)t}$ , which is the right answer (see Eq. (5)). Without this extra term we would be lead to the conclusion that for  $v = 0$ , i.e.  $\langle \eta \rangle = 0$ , the average (not typical!)  $W(t)$  does not grow, which is wrong.

The other way to get this extra term in the equation for  $W$  is to start with the well known Langevin equation of motion for  $h(t) = \ln W(t)$  describing a usual random walk with a drift:

$$\frac{dh(t)}{dt} = v + \tilde{\eta}(t), \tag{7}$$

where again  $\langle \tilde{\eta}(t) \rangle = 0$ , and  $\langle \tilde{\eta}(t)\tilde{\eta}(t') \rangle = D\delta(t - t')$ . To derive the equation of motion for  $W(t) = e^{h(t)}$  one has to do the change of variables as for usual partial differential equations. But in addition to this one has to add the “Ito term” [5] given by  $(D/2)(\partial^2 W/\partial h^2)$ , which is a formal prescription of Ito calculus. With this nontrivial correction one recovers the equation of motion (6). So in the language of SDE the difference between the typical ( $v$ ) and the average ( $v + D/2$ ) growth rates of  $W(t)$  in the multiplicative random walk is a direct consequence of the Ito term, appearing after the change of variables from  $h$  to  $W$  in Eq. (7).

### 2.3. Multiplicative random walk in the presence of a lower wall

Much attention was devoted recently [2,3] to the analysis of the problem of “multiplicative random walk, repelled from zero”. In the context of economics it was first

introduced by Solomon et al. [2]. In a simplest case one has a multiplicative random walk with a Gaussian random variable  $\eta$ , having a negative average  $v = \langle \eta \rangle < 0$ , and the dispersion  $D$ . In other words the typical value of  $W(t)$  exponentially decreasing in time, while its average may or may not grow in time depending on the sign of  $v + D/2$ . In addition to this one has an “external force”, pushing  $W(t)$  up and preventing it from falling below some predetermined constant. This external influence, which will be referred to as “lower wall”, should not significantly affect the dynamics for large  $W$ . One way to introduce a lower wall is to add an additional positive “source” term  $b$  into the RHS of Eq. (6). Eqs. (6) and (7) now become

$$\frac{dW(t)}{dt} = (v + D/2)W(t) + \eta(t)W(t) + b, \quad (8)$$

$$\frac{dh(t)}{dt} = v + \eta(t) + b \exp(-h). \quad (9)$$

As we see the lower wall in Eq. (9) has a property of being “short-ranged” in  $h$ -space, i.e. its contribution to the SDE for  $h(t)$  can be neglected for large positive  $h$ . But for negative  $h$  the strength of the wall grows exponentially and compensates the negative drift already at  $h = -\ln(|v|/b)$ . It is easy to convince oneself that this stochastic process eventually reaches a stationary state, characterized by a stationary probability distribution  $P(h)$ . In this stationary state the negative drift of  $h(t)$  is precisely balanced with diffusion combined with repulsion from the lower wall.

In the literature on this subject one encounters many different realizations of the lower wall mechanism. For instance, one can introduce a more general term  $bW^\delta$  into the RHS of Eq. (6) [6,7]. In the equation for  $h$  this term becomes  $b e^{(\delta-1)h}$ , which for any  $\delta < 1$  describes an exponential lower wall qualitatively similar to Eq. (9). Indeed, the “source” term in Eq. (8) is just a particular example of this more general term with  $\delta = 0$ . On the other hand, the terms with  $b < 0$  and  $\delta > 1$  describe an “upper wall”, preventing  $h$  from becoming too big. In this case, in order for a stationary state to exist one needs a positive drift of  $h$  pushing it up against the wall. In [2] the lower wall is introduced “by hand”: in their simulations the authors simply do not allow  $h(t)$  to fall below a predetermined constant  $h_{min}$ . In other words,  $h(t+1) = \min(h(t) + \eta(t), h_{min})$ . Such “infinitely hard lower wall” can be described by a term  $bW^\delta$  with very large negative  $\delta$ . Finally, Sornette and Cont [3] consider a case when the constant  $b$  itself can depend on time obeying a deterministic and/or stochastic dynamics. Except for pathological cases, where typical  $b(t)$  exponentially grows or decays in time, it does not qualitatively change the results, compared to a time-independent lower wall [2].

An interesting feature of a multiplicative random walk with a lower wall is that it generically gives rise to a power law tail in the distribution of  $W$  in the stationary state. We proceed by reviewing various derivations of this result found in recent literature [2,3]. As was explained above, the lower wall’s only purpose is to make the process stationary by pushing the variable up whenever it becomes too small. The drift due to the wall can always be neglected for large enough  $h$ . In the region, where this approximation is justified one can write a Fokker–Planck equation, taking into account

only the multiplicative part of the process, equivalent to diffusion with a drift in the  $h$ -space. The stationary solution of the Fokker–Planck equation should satisfy

$$-v \frac{\partial P(h)}{\partial h} + \frac{D}{2} \frac{\partial^2 P(h)}{\partial h^2} = 0.$$

It is easy to see that  $P(h) = A \exp(2vh/D)$  is indeed a solution. Since  $v < 0$ , it exponentially decays for positive  $h$ . The deviations from this form start to appear only at low  $h$ , where the presence of lower wall cannot be neglected. This “Boltzmann” tail of the distribution of  $h$  corresponds to a power law tail of distribution of  $W = e^h$ :  $P(W) = AW^{-1+2v/D}$ . The exponent of this power law tail

$$\tau = 1 - 2v/D = 1 + 2|v|/D \tag{10}$$

is greater than 1, so that there are no problems with normalization. In case of a lower wall of the form  $be^{-h}$  (see Eq. (9)) one can write an analytic solution of the Fokker–Planck equation valid for any  $h$ . It is the Boltzmann distribution with a Hamiltonian  $H(h) = be^{-h} - vh$  and temperature  $T = D/2$ , i.e.  $P(h) = A \exp[(-2be^{-h}/D + 2vh/D)]$ , or  $P(W) = A \exp(-2b/DW)W^{-1+2v/D}$ . The normalization constant  $A$  is given by  $A = (2b/D)^{-2v/D} / \Gamma(-2v/D)$ .

Eq. (10), expressing the exponent of the power law tail of  $P(W)$  in terms of  $v$  and  $D$ , is valid only for the case of Gaussian distribution  $\pi(\eta)$ . Indeed, in its derivation we employed a stochastic differential equation approach, which is restricted to Gaussian noise. It is instructive to derive an equation, giving the value of  $\tau$  for a general  $\pi(\eta)$ . It was first done by Kesten in [8] and recently brought to the attention of physics community in [3]. Again, the formula holds for any multiplicative process with a negative average drift ( $\langle \eta \rangle < 0$ ) and a lower wall, the effect of which can be neglected for large  $W$ . We assume that the process has already reached a stationary state, characterized by a stationary distribution  $P(W)$ . For sufficiently large  $W$ , so that one can neglect the effect of the wall, the stationarity imposes the following functional equation on  $P(W)$ :  $P(W) = \int_{-\infty}^{+\infty} \pi(\eta) d\eta \int_0^{+\infty} P(W') \delta(W - e^\eta W') dW' = \int_{-\infty}^{+\infty} d\eta \pi(\eta) e^{-\eta} P(We^{-\eta})$ . Assuming that the solution has a power law tail  $P(W) \sim W^{-\tau}$  one finds  $\int_{-\infty}^{+\infty} d\eta \pi(\eta) e^{\eta(\tau-1)} = 1$ . In other words  $\tau$  is given by a solution of

$$\langle e^{\eta(\tau-1)} \rangle = 1. \tag{11}$$

The obvious solution  $\tau = 1$  should be rejected because the distribution function is not normalizable in this case. In short, we are looking for a solution with  $\tau > 1$ . Let us define  $\Lambda(\tau) = \langle e^{\eta(\tau-1)} \rangle$ . Since  $d\Lambda(1)/d\tau = \langle \eta \rangle < 0$ , but  $d^2\Lambda(\tau)/d\tau^2 > 0$  one has at most one such a solution. In fact, if the distribution of  $p(\eta)$  is not restricted to  $\eta < 0$ , for  $\tau \rightarrow +\infty$  one has  $\Lambda(\tau) \rightarrow +\infty$  and the solution is guaranteed by the continuity of  $\Lambda(t)$ . Only in the situation when  $\eta$  is always negative, the region of large  $W$  is absolutely inaccessible, and no power law tail at large  $W$  is feasible. Using Eq. (4), one can check that for a Gaussian distribution Eq. (11) predicts  $\tau = 1 - 2v/D$  in agreement with Eq. (10).

## 2.4. Interpretation of $W(t)$ as a fluctuating stock capital

In what follows we will stick to the following “realization” of the random multiplicative process: we interpret  $W(t)$  as the capital (or wealth, hence the notation) that a single investor has in some stock. The price of the share of this stock  $p(t)$  undergoes a random multiplicative process  $p(t+1) = e^{\eta(t)} p(t)$ , and if the investor keeps a fixed number  $K$  of shares without selling or buying this stock, his capital  $W(t) = Kp(t)$  follows these price fluctuations. Later on we will consider models, where the investor at each time step will sell some stock and buy another. We assume that volumes of such transactions are sufficiently small, so that they have no influence on the market price fluctuations. Hence our assumption that  $\eta(t)$  and  $W(t)$  are uncorrelated.

The lesson one derives from the above properties of multiplicative random walk is that if the investor keeps all his money in just one stock it is the *typical* growth rate  $\langle \eta \rangle$ , he should be concerned about. In majority of realizations his capital grows at *typical* rate and he cannot directly take advantage of a bigger *average* growth rate  $\ln \langle e^\eta \rangle$ . There are situations when the typical growth rate is negative, i.e. the stock price is going down, while the fluctuations are strong enough to make the average rate positive. The question we are going to address in this manuscript is how one can still exploit this *average* growth rate by investing and actively managing a portfolio composed of  $N$  stocks.

## 3. Ensemble of $N$ stocks without redistribution

The first problem we are going to consider is: what is the *typical* growth rate of the capital invested in an ensemble of  $N$  stocks if one is not allowed to sell one of them and reinvest the money into another. In the following we assume that the price  $p_i(t)$  of a share of each stock undergoes a multiplicative random walk, independent of price fluctuations of other stocks. In other words, one time step logarithmic price increments  $\eta_i(t)$  are uncorrelated not only at different times, but also for different stocks at a given time. The validity of this approach for the real stock market lies beyond the scope of this work. For simplicity of final expression in this section we will restrict ourselves to the situation when  $\eta_i$  for each of the stocks are Gaussian variables with zero mean ( $\langle \eta_i \rangle = \nu = 0$ ) and the same dispersion  $D = \langle \eta_i \rangle$ . Initially the capital is equally distributed between all stocks. We assume that the starting capital in each stock is equal to  $1/N$ , so that the total capital is equal to 1. The typical value of the total capital  $(W_{tot}(t))_{typ} = (\sum_{i=1}^N W_i(t))_{typ}$  will then grow in time. From the results of the previous section one concludes that  $\langle W_i(t) \rangle = e^{Dt/2}$  and  $\langle W_i(t)^2 \rangle - \langle W_i(t) \rangle^2 = e^{2Dt} - e^{Dt}$ . One can safely replace the sum of  $N$  variables with their average as long as  $((\langle W_i(t)^2 \rangle - \langle W_i(t) \rangle^2)/N)^{1/2} \ll \langle W_i(t) \rangle$ . Therefore, at short times, when  $Dt \ll \ln N$ , one indeed enjoys the average growth rate:  $(W_{tot}(t))_{typ} = e^{Dt/2}$ . At later times, however, the typical value of the capital starts to fall below the average value (i.e. average value over infinitely many realizations). To determine this slower

growth of typical value quantitatively one has to approach the problem from a different end. At late times the value of the total capital is mainly determined by the capital accumulated in the most successful stock, i.e.  $W_{tot}(t) \simeq W_{\max} \equiv \max_{i=1,N} W_i(t)$ . The extremal statistics theory [9] readily gives the typical value of the  $W_{\max}$  by requiring that  $1/N = \text{Prob}(W > W_{\max}) = \text{Prob}(\ln W > \ln W_{\max}) \sim \exp(-\ln^2 W_{\max}/2Dt)$ . With exponential precision one gets  $W_{\max} \sim e^{\sqrt{2Dt \ln N}}$ . Our approximation that  $W_{tot}(t) \simeq \max_{i=1,N} W_i(t)$  is good only if the second maximal  $W$  (we denote it as  $W_{\max}^{(2)}(t)$ ) is much smaller than the maximal one. Following the same arguments as before one concludes to find the typical value of  $W_{\max}^{(2)}(t)$  one needs to solve  $\text{Prob}(W > W_{\max}^{(2)}) = 2/N$ . This results in  $W_{\max}^{(2)} \sim e^{\sqrt{2Dt(\ln N - 2)}}$ . One easily confirms that the approximation of the whole sum with its biggest element makes sense if  $Dt \gg \ln N$ , which is a complementary condition to the “average” growth at small times. Therefore, we conclude that

$$(W_{tot}(t))_{typ} = e^{Dt/2} \quad \text{for } t \ll \ln N/D, \quad (12)$$

$$(W_{tot}(t))_{typ} = e^{\sqrt{2Dt \ln N}} \quad \text{for } t \gg \ln N/D. \quad (13)$$

Since growth proportional to  $\sqrt{t}$  is slower than linear in  $t$  one concludes that no matter how big is your  $N$  your asymptotic growth of your total capital is still determined by the “typical” growth rate  $v = \langle \eta \rangle$  (equal to zero in the case considered above) of a single stock.

If one wants to exploit the “average” growth rate for a period of time  $T$  and then sell the stocks one needs to take an exponentially large ensemble of stocks  $N > e^{DT}$ .

#### 4. Ensemble of $N$ stocks with redistribution

The case of “non-interacting” stocks, considered in the previous section, can be also called the case of a “lazy investor”. Indeed, initially the investor puts equal capital in  $N$  stocks and leaves them as they are. He never sells or buys stocks. No wonder that very soon he can no longer expect to get an *average* rate of return on his investment and has to settle for smaller *typical* growth rate. Now we are going to consider the case of an *active* investor who after each time step redistributes his capital between stocks according to some simple rule. One may naively think that by selling unsuccessful stocks with small  $W_i$  and reinvesting the money into successful stocks with large  $W_i$  one may do better. In reality the answer is precisely the opposite: one needs to sell some of the most successful stocks and reinvest the money into the least successful stocks. Selling only small number of shares of the most successful stocks (i.e. ones which are currently overpriced) and reinvesting this money into the least successful stocks (i.e. ones which are currently underpriced) makes a huge difference:  $\ln W_i$  for underpriced stocks goes up significantly, while  $\ln W_i$  for overpriced stocks does not go down as much. As we will show such a “charity” between stocks bootstraps the typical growth rate of the capital, so that  $\ln(W_{tot}(t))_{typ}$  at all times has a growth rate bigger than a *typical* growth rate of a single stock. For large  $N$  this rate quickly approaches

the average growth rate given by  $\ln \langle e^\eta \rangle$  (equal to  $D/2$  for the Gaussian distribution of  $\eta$  with zero mean). This growth rate serves as a theoretical maximum of all possible growth rates achievable by simple redistribution of funds.

#### 4.1. Problem with redistribution in the discrete time approach

We start with the simplest strategy for redistribution of the capital. Under this strategy at each time step the investor calculates the current value of average capital per one stock  $\overline{W}(t) = (1/N) \sum_{i=1, N} W_i(t)$ . The capital is redistributed between the stocks according to the rule  $\overline{W}_i \rightarrow W_i - \lambda(W_i - \overline{W})$ . For positive  $\lambda$  it means that “overpriced” stocks with  $W_i(t) > \overline{W}(t)$  lose a fraction of their capital in favor of the “underpriced” ones with  $W_i(t) < \overline{W}(t)$ . The extremal case of  $\lambda = 1$  corresponds to the equal redistribution of the capital after each time step. The stock price changes during the next discrete time interval. As a result the capital invested in each stock is multiplied by the random factor  $e^{\eta_i(t)}$ . The complete change of each stock’s capital after one time step is given by

$$W_i(t+1) = e^{\eta_i(t)} [(1-\lambda)W_i(t) + \lambda\overline{W}(t)]. \quad (14)$$

One can recognize the above model can be interpreted as the Directed Polymer model in  $N$  dimensions, with mean field (fully connected) interactions [10]. The role Laplacian is played by  $\overline{W}(t) - W_i(t) = (1/N \sum_{j=1, N} W_j(t)) - W_i(t)$ . It is convenient to introduce a new set of rescaled variables  $s_i(t) = W_i(t)/\overline{W}(t)$ . The sum of  $s_i$  is always equal to  $N$ , which sets a theoretical cutoff equal to  $N$  to a value of individual  $s_i$ . One can rewrite Eqs. (14) in the following form:

$$s_i(t+1) = \frac{\overline{W}(t)}{\overline{W}(t+1)} e^{\eta_i(t)} [(1-\lambda)s_i(t) + \lambda], \quad (15)$$

$$\frac{\overline{W}(t+1)}{\overline{W}(t)} = \frac{\sum_{i=1, N} e^{\eta_i(t)} [(1-\lambda)s_i + \lambda]}{N}. \quad (16)$$

As we will confirm later, the dynamics of  $\overline{W}(t)$  can be approximated as a random multiplicative process, where the multiplication factor  $\Gamma(t) = \sum_{i=1, N} e^{\eta_i(t)} ((1-\lambda)s_i + \lambda)/N$  has only small fluctuations around its average value. We will indeed demonstrate that  $\Gamma(t) = \langle \Gamma \rangle + \delta\Gamma(t)$ , where  $|\delta\Gamma(t)| \sim N^{-\alpha/2}$ . It means that for large  $N$  to a good approximation one can disregard the fluctuations of  $\overline{W}(t+1)/\overline{W}(t)$  while trying to solve Eq. (15). The average value of this ratio is easily calculated and is equal to  $\langle e^\eta \rangle$  (one has to recall that  $\sum_{i=1, N} s_i = N$ ). In this approximation the equations of motion for  $s_i$  decouple and allow for exact solution. These *mean-field* equations are:

$$s_i(t+1) = \frac{e^{\eta_i(t)}}{\langle e^\eta \rangle} [(1-\lambda)s_i(t) + \lambda]. \quad (17)$$

Similar equation of motions were recently studied by Cont and Sornette [3] and Solomon and Levy [2] and were shown to give rise to a stationary distribution of

$s$  having a power law tail for large  $s$ . One has to keep in mind that the definition of  $s$  in our problem introduces a natural cut off to this tail as  $s \leq N$ , so it is only for large  $N$  that one has a chance to see the effect of this power law or measure this power law numerically.

The stationary distribution  $P(s)$  is conserved by dynamics. Therefore, it should satisfy the following functional equation:

$$P(s) = \int d\eta \pi(\eta) P\left(\frac{s}{R(\eta)} - \frac{\lambda}{1-\lambda}\right) / R(\eta), \tag{18}$$

where  $R(\eta) = (1 - \lambda)e^\eta / \langle e^\eta \rangle$ . Using this equation one can easily verify that indeed  $\langle s \rangle = \int sP(s) ds = 1$ , which is to be expected since  $\sum_{i=1, N} s_i = N$ . Assuming that  $P(s)$  has a power law tail of the form  $As^{-\tau}$ , and substituting it to the functional equation (18) one gets the self consistency condition for the exponent  $\tau$ :  $\int d\eta \pi(\eta) R(\eta)^{\tau-1} = 1$ , or

$$\frac{\langle e^{\eta(\tau-1)} \rangle^{1/(\tau-1)}}{\langle e^\eta \rangle} = \frac{1}{1-\lambda}. \tag{19}$$

For a general distribution  $\pi(\eta)$  this equation cannot be solved analytically. All one can deduce is that for a weak coupling  $\lambda \ll 1$  the solution exists and is approximately given by  $\tau = 2$ . That means that for a weak coupling one always has  $P(s) \sim 1/s^2$ ! For a case of Gaussian distribution of  $\eta$  the analytic expression for  $\tau$  can be easily obtained from Eq. (4) and is given by

$$\tau = 2 - \frac{2 \ln(1 - \lambda)}{D}. \tag{20}$$

In Fig. 1 we present the results of simulations of the model with  $N = 10000$ . The measured power law exponent is in excellent agreement with the above theoretical prediction.

Our ultimate goal is to determine  $\overline{W(t)}_{typ}$  as a function of  $t$ . The Eq. (16) states that at each time step  $\overline{W(t)}$  is multiplied by  $\Gamma(t) = \sum_{i=1, N} e^{\eta_i(t)} ((1 - \lambda)s_i + \lambda) / N$ . One can show that  $\Gamma(t)$  at different time steps are uncorrelated. One can also disregard possible correlations between the value of  $\overline{W(t)}$  and  $\Gamma(t)$  at the same time step. Then the behavior of  $\overline{W(t)}$  is nothing else but a multiplicative random walk studied in Section 1. The typical value of  $\overline{W(t)}$  grows as  $(\overline{W(t)})_{typ} = e^{t \langle \ln \Gamma \rangle}$ , while its average value grows as  $e^{t \ln \langle \Gamma \rangle} = e^{t \ln \langle e^\eta \rangle} = \langle e^{\eta t} \rangle$ .

We will proceed by demonstrating that for any  $\lambda > 0$  the typical and average growth rates of  $W(t)$  differ by  $O(N^{-\alpha})$ . For  $\langle \ln \Gamma \rangle$  one has the exact expression:

$$\langle \ln \Gamma \rangle = \ln \langle e^\eta \rangle \left\langle \ln \left( 1 + \frac{1}{N} \sum_{i=1}^N \chi_i [(1 - \lambda)s_i + \lambda] \right) \right\rangle \tag{21}$$

where we introduced the notation  $\chi_i = e^{\eta_i} / \langle e^\eta \rangle - 1$ . Expanding the second logarithm for large  $N$ , we get to leading order:

$$\langle \ln \Gamma \rangle \simeq \ln \langle e^\eta \rangle - \frac{1}{2N^2} \sum_{i=1}^N \langle \chi_i^2 \rangle \langle [(1 - \lambda)s_i + \lambda]^2 \rangle,$$

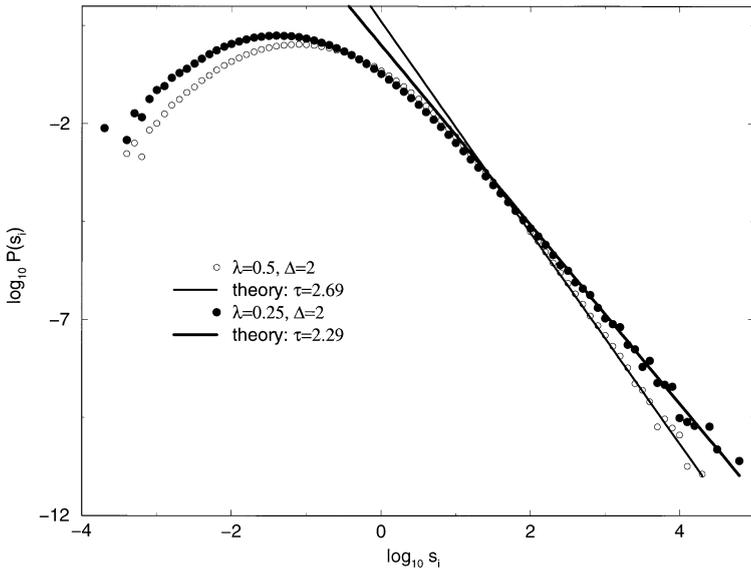


Fig. 1. The distribution of capital fractions  $s_i = W_i/\overline{W}$  for  $\lambda = 0.25, 0.5$ , and gaussian  $\pi(\eta)$  with  $D=2$ , and  $v=0$  in a system of size  $N=10000$ . The solid lines are the theoretical predictions (20) for a power law exponent  $\tau$  of the tail of this distribution.

where we used the fact that  $\chi_i(t)$  are uncorrelated at different  $i$ 's. Therefore,  $\langle \chi_i(t)\chi_j(t') \rangle = \tilde{D}\delta_{i,j}\delta_{t,t'}$ , where  $\tilde{D} = \langle e^{2\eta} \rangle / \langle e^\eta \rangle^2 - 1$ . The fact that these variables are uncorrelated at different times proves that indeed  $\overline{W}(t)$  undergoes a multiplicative random walk. The last step is to estimate  $\sum_{i=1,N} s_i^2$ . To do this we need to recall our results for the stationary distribution  $P(s)$ . If the exponent  $\tau$  of the power law tail of this distribution is larger than 3,  $\langle s^2 \rangle$  is finite,  $\sum_{i=1,N} s_i^2 = N\langle s^2 \rangle$  and one immediately gets  $\langle \ln \Gamma \rangle = \ln \langle \Gamma \rangle - A/N$ , where  $A = [(1-\lambda)^2 \langle s^2 \rangle + \lambda^2 + 2(1-\lambda)]\tilde{D}$ . In reality this is not hundred percent true. Indeed, expanding the logarithm in Eq. (21) we stopped at the first order. In the presence of power law tails in  $P(s)$  the validity of this approximation is in doubt because the higher-order terms involve the sum of powers  $s_i^k$  with  $k > 2$ . For large enough  $k$  such powers are known to diverge as some power of  $N$ . It can be shown that for *very large*  $N$  they would dominate the scaling with respect to  $N$ . Such crossover was indeed observed in simulations. In Fig. 2 we present the results of the simulations of our model with  $\lambda=0.1$ ,  $D=0.1$ , which corresponds to  $\tau=4.1$ . Indeed, we observe that for  $N \rightarrow \infty$ , the difference between the average and typical growth rates of the total capital,  $v_{avg} - v_{typ}(N) = \ln \langle \Gamma \rangle - \langle \ln \Gamma \rangle$ , approaches zero. This approach starts as  $A/N^\alpha$  with  $\alpha=1$ , but at larger  $N$  a deviation towards smaller  $\alpha$  can be noticed.

For  $2 < \tau < 3$  the second moment of  $s$  diverges. This means that one should be more careful in estimating  $\sum_{i=1,N} s_i^2$ . The apparent divergence of the integral  $\int s^2 P(s) ds$  should not be taken too seriously, since we are dealing with a finite sample of variables  $s$  restricted by  $\sum s_i = N$ . Even in the worst case if only one  $s_i$  is nonzero (and equal

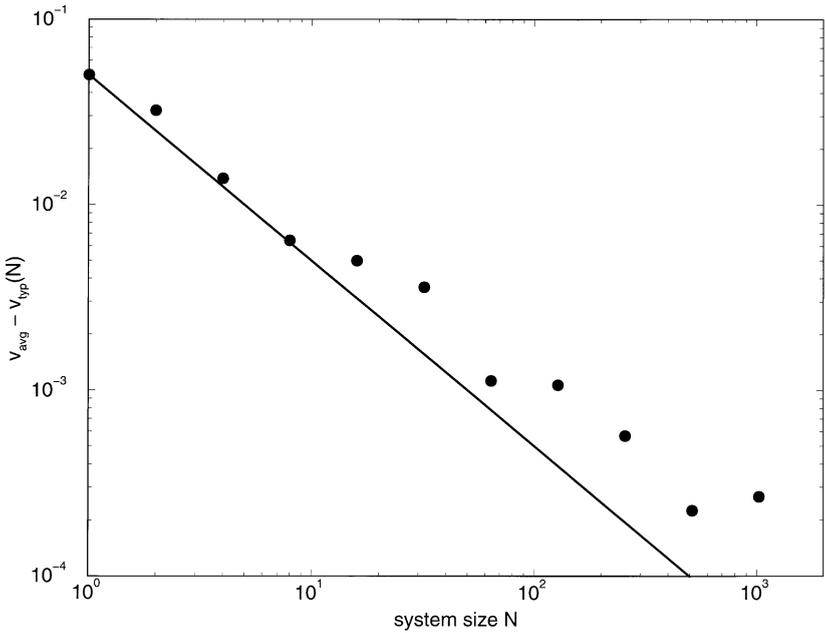


Fig. 2. The difference between the average growth rate of the capital  $v_{avg} = D/2$  and its typical growth rate  $v_{typ}(N)$  as a function of the number of assets  $N$ . The parameters of the model are  $\lambda = 0.1$ ,  $D = 0.1$ ,  $v = 0$ . The solid line indicates the theoretical prediction  $A/N$ . The crossover towards smaller  $\alpha$  is clearly seen for large  $N$ .

to  $N$  by normalization) the sum  $\sum_{i=1,N} s_i^2 = N^2$ . In all situations when the integral  $\int s^k P(s) ds$  diverges the sum of a finite sample is dominated by the largest element. One can estimate this largest  $s$  by requiring  $Prob(s > s_{max}) = s_{max}^{1-\tau} = 1/N$ . Therefore, the typical value of the largest  $s_i$  is given by  $s_{max} \sim N^{1/(\tau-1)}$ . Since  $\tau \geq 2$  this value is always less than  $N$  – the maximal possible  $s$ . Then  $\sum_{i=1,N} s_i^2 \simeq s_{max}^2 \sim N^{2/(\tau-1)}$ . Now the expression for the  $\langle \ln \Gamma \rangle$  becomes  $\langle \ln \Gamma \rangle = \ln \langle \Gamma \rangle - A'/N^{-2+2/(\tau-1)}$ , with  $A' \sim (1 - \lambda)^2 \tilde{D}$ .

#### 4.2. Problem with redistribution in continuous time approach

Similar results can be obtained in the continuous time limit of Eq. (14). In order to derive the stochastic partial differential equation corresponding to Eq. (14) we assume that time is discretized  $t = n \Delta t$  in units  $\Delta t$  and we take  $\lambda = \lambda_c \Delta t$ ,  $v = v_c \Delta t$ , and  $D = D_c \Delta t$ . In all our future formulas we drop the subscript  $c$  in  $\lambda_c$ ,  $v_c$ , and  $D_c$  of the continuous model. However, one should keep in mind that we recover the continuous limit by making parameters  $\lambda, v$ , and  $D$  of a discrete model very small, keeping their ratio fixed.

In the limit  $\Delta t \ll 1$  the Eq. (14) becomes a stochastic differential equation

$$\partial_t W_i(t) = \lambda(\bar{W} - W_i) + (v + D/2) W_i + W_i \tilde{\eta}_i(t). \tag{22}$$

Here as in Section 1 we introduced the continuous-time stochastic force  $\tilde{\eta}_i(t) = \eta_i(t)/\Delta t - v$ , and used  $e^{\eta_i} = 1 + \eta_i + \eta_i^2/2 + \dots \simeq 1 + \tilde{\eta}_i \Delta t + (v + D/2) \Delta t + O(\Delta t^{3/2})$ . It is important to point out here that such a continuous time formulation is only meaningful if  $\eta_i(t)$  is a Gaussian noise. Only in this case Eq. (22) can be regarded as a Langevin equation [5]. Usually the assumption of a Gaussian noise is motivated by the fact that for a continuous time process, the stochastic force  $\tilde{\eta}_i dt$  acting on a small interval  $\Delta t$  can be thought of a sum of infinitely many infinitesimal contributions. The central limit theorem then ensures that  $\tilde{\eta}_i \Delta t$  is Gaussian. For processes with additive noise, this assumption is reasonable also for discrete time processes. For multiplicative processes the deviations from the central limit theorem becomes of concern since the tails of the distributions are probed by the process. Therefore, we shall assume in this section that  $\tilde{\eta}_i$  is Gaussian.

Under this assumption, we shall be able to derive the full probability distribution of the  $W_i$  in the limit  $N \rightarrow \infty$ . It is again convenient to use the variables  $s_i(t) = W_i(t)/\overline{W}(t)$ . Using Ito calculus, one readily finds

$$\partial_t s_i = \lambda(1 - s_i) - \frac{D}{N} s_i (s_i - \overline{s^2}) + s_i \tilde{\eta}_i - \overline{s \tilde{\eta}} s_i \quad (23)$$

where we used the notation  $\overline{f} = (1/N) \sum_j f_j$ . Note that  $\sum_i s_i = N$  and, consistently,  $\sum_i \partial_t s_i = 0$ . We shall adopt a self consistent mean field approach, valid in the  $N \rightarrow \infty$  limit, in which we substitute averages over  $i$  with statistical averages:  $\overline{f} \cong \langle f \rangle$ . Within this approximation, the term  $\overline{s \tilde{\eta}} \cong \langle s \tilde{\eta} \rangle = 0$  can be neglected. If we introduce

$$\tau = \frac{2\lambda}{D} - \frac{2}{N} \langle s^2 \rangle \cong \frac{2\lambda}{D} - \frac{2}{N} \overline{s^2} \quad (24)$$

as a constant to be determined later self-consistently, Eq. (23) becomes an equation for  $s_i$  only, which does not involve  $s_j$  for  $j \neq i$  explicitly. We know [5] that, for a Langevin equation of the form

$$\partial_t s = -s^2 \frac{dV(s)}{ds} + s \tilde{\eta},$$

the associated Fokker–Planck equation yields the asymptotic distribution  $P(s) \sim e^{-2V(s)/D}$ . Recasting Eq. (23) into this form, we find

$$V(s) = \frac{\lambda}{s} + \frac{\tau D \ln s}{2} + \frac{Ds}{N},$$

from which

$$P(s) = \mathcal{N} \exp \left[ -\frac{2\lambda}{Ds} - \frac{2}{N} s \right] s^{-\tau}. \quad (25)$$

Note the emergence of a power law behavior in  $P(s)$ , which is however cut off by the second term in the exponential. This is physically meaningful, since  $s \leq N$  must hold, with  $s = N$  occurring when the whole capital  $N\overline{W}$  is invested in a single stock. The value  $\tau$  of the power law decay is determined self-consistently from Eq. (24)

performing the average on the distribution in Eq. (25). A further requirement which our approach imposes on  $P(s)$  is that  $\bar{s} \cong \langle s \rangle = 1$ . It is not possible to compute exactly these averages, however, it is possible to perform a large  $N$  expansion. Indeed if we set

$$Z(\mu) = \int_0^\infty ds \exp \left[ -\frac{2\lambda}{Ds} - \mu s \right] s^{-\tau}$$

then, clearly,  $\langle s \rangle = -\partial_\mu \ln Z(\mu)|_{\mu=2/N}$  and  $\langle s^2 \rangle = \partial_\mu^2 \ln Z(\mu)|_{\mu=2/N} + \langle s \rangle^2$ . Therefore, evaluating the small  $\mu$  expansion of  $Z(\mu)$  we can compute the first two moments of  $s$  and impose self-consistency. However  $Z(\mu)$  has a non-analytic expansion around  $\mu=0$ , since derivatives  $\partial_\mu^n Z(\mu)$  diverge at  $\mu=0$  for  $n \geq \tau$ . For  $\lambda > D/2$ , the first two derivatives exist. The equation  $\langle s \rangle = 1$  then allows us to compute  $\tau$  together with its leading correction:

$$\tau \cong 2 + \frac{2\lambda}{D} - \frac{\lambda((D/2)^2 - \lambda D/2 + \lambda^2)}{(D/2)^2(D/2 - \lambda)} \frac{4}{N} \quad \text{for } D/2 < \lambda. \tag{26}$$

Eq. (24) then turns out to be automatically satisfied, which is a reassuring check of self-consistency. Note that Eq. (20) derived previously, exactly reduces to Eq. (26) with  $\lambda \leq 1$ . For  $\tau < 3$  the second derivative of  $Z(\mu)$  does not exist at  $\mu=0$ . The second term in Eq. (26) changes, but the leading term remains the same:

$$\tau \cong 2 + \frac{2\lambda}{D} - \frac{2\lambda/D + 1}{\Gamma(2\lambda/D + 1)} \int_0^\infty dx \frac{e^{-x} - 1 + x}{x^{2+2\lambda/D}} \left[ \frac{4\lambda}{DN} \right]^{2\lambda/D}.$$

The average growth rate of the capital  $N\bar{W}$  is obtained summing Eq. (22) over  $i$  and dividing by  $N$ :

$$\partial_t \bar{W}(t) = (v + D/2 + \overline{s\tilde{\eta}}) \bar{W}.$$

The solution to this equation

$$\begin{aligned} \bar{W}(t) &= \bar{W}(0) \exp \left[ (v + D/2)t + \int_0^t \overline{s\tilde{\eta}}(t') dt' \right] \\ &\cong \bar{W}(0) e^{(v+D/2)t} \end{aligned}$$

implies that the growth rate of the average is, to leading order in  $N$ , equal to the growth rate of the average  $v + D/2$ .

### 4.3. Parallels to directed polymers in random media

In conclusion we would like to point out that the stochastic differential equation (22) has a finite-dimensional analogue, which was much studied over the past decade. Indeed, the term  $\lambda(\bar{W} - W_i)$  is nothing else but a fully connected (infinite dimensional)

variant of discrete Laplacian. In finite dimensions this term becomes  $\lambda \Delta W_i = \lambda (\sum_{nn} W_{nn}/2d - W_i)$ . In the spatial continuous limit the Eq. (22) becomes

$$\partial_t W(x, t) = \lambda \Delta W(x, t) + (v + D/2)W(x, t) + \eta(x, t)W(x, t), \quad (27)$$

which can be easily recognized as the equation for the partition function of directed polymer in random media [10]. The change of variables  $h = \ln W$  maps this equation to the so-called KPZ equation [11]:

$$\partial_t h(x, t) = \lambda(\Delta h(x, t) + |\nabla h(x, t)|^2) + vh(x, t) + \eta(x, t). \quad (28)$$

In our infinite-dimensional (fully connected) model we found that  $P(W)$  has a power law behavior for large  $W$ . In finite dimensions, at least below the upper critical dimension  $d_c$  (whose very existence is still under debate), this seems not to be the case. Indeed numerical simulations show that, at least up to  $d = 3 + 1$  [12] the distribution of  $h = \ln W$  has not a pure exponential, but rather stretched exponential behavior. We conjecture that the power law behavior of  $P(W)$  in the model studied in this manuscript is an artifact of the peculiar long range interaction, where each site is coupled to any other site.

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